Math 4130/5130 Homework 6

3.B # 5 Give an example of a linear map $T : \mathbb{R}^4 \to \mathbb{R}^4$ such that range$(T) = (T)$.

Let $e_1, \dots, e_4$ denote the standard basis vectors for $\mathbb{R}^4$. Define T by

$$Te_1 = e_3, \quad Te_2 = e_4, \quad Te_3 = 0, \quad Te_4 = 0.$$

3.B # 6 Prove that there does not exist a linear map $T : \mathbb{R}^5 \to \mathbb{R}^5$ such that range$(T) = (T)$.

Use the fact that $5 = dim(\mathbb{R}^5) = dim(null(T)) + dim(range(T))$. If $range(T) = null(T)$ then their dimensions would be the same and we would have 5 as an even number.

3.B # 11 Suppose $S_1, \dots, S_N$ are injective linear maps such that $S_1 S_2 \cdots S_N$ makes sense. Prove that $S_1 S_2 \cdots S_N$ is injective.

By Theorem 3.16 of the text, a linear map T is injective if and only if the null space of T is 0.

It's probably cleanest to use induction and to flip the indexing, asking if $S_N \cdots S_2 S_1$ is injective. In case $N = 1$ the map S_1 is injective by assumption. Suppose the composition of at most N injective maps is injective and

$$S_{N+1} S_N \cdots S_2 S_1 v = 0.$$

Since S_{N+1} is injective, $S_N \cdots S_2 S_1 v = 0$. But by the induction hypothesis, $S_N \cdots S_2 S_1$ is injective, so $v = 0$ and $S_{N+1} S_N \cdots S_2 S_1$ is injective.

3.B # 12 Suppose that $\mathbb{V}$ is finite dimensional and that $T \in \mathcal{L}(\mathbb{V}, \mathbb{W})$. Prove that there exists a subspace U of $\mathbb{V}$ such that $U \cap \mathrm{null}(T) = \{0\}$ and $\mathrm{range}(T) = \{Tu : u \in U\}$.

First pick a basis $(w_1, \dots, w_J)$ for $\mathrm{null}(T)$. Extend this to a basis

$$(w_1, \dots, w_J, u_1, \dots, u_K)$$

for $\mathbb{V}$. Define $U = \mathrm{span}(u_1, \dots, u_K)$.

First notice that if $v \in U \cap \mathrm{null}(T)$, then

$$v = \sum_k \alpha_k u_k = \sum_j \beta_j w_j.$$

Rewriting this, we have

$$0 = \sum_k \alpha_k u_k - \sum_j \beta_j w_j,$$

and the linear independence of $(w_1, \ldots, w_J, u_1, \ldots, u_K)$ implies $\alpha_k = \beta_j = 0$. Thus $v = 0$, and $U \cap \text{null}T = \{0\}$.

Finally, any $v \in \mathbb{V}$ can be written as

$$v = \sum_j \beta_j w_j + \sum_k \alpha_k u_k,$$

so

$$Tv = \sum_j \beta_j T w_j + \sum_k \alpha_k T u_k = T(\sum_k \alpha_k u_k),$$

showing that $\text{range}T = \{Tu : u \in U\}$.

3.C # 2 Find bases for $\mathcal{P}_3$ and $\mathcal{P}_2$ such that the matrix for $p \to p'$ is

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Let the basis for $\mathcal{P}_3$ be $z^3, z^2, z, 1$ and the basis for $\mathcal{P}_2$ be $3z^2, 2z, 1$.

3.C # 3 Suppose $\mathbb{V}$ and $\mathbb{W}$ are finite dimensional and $T \in \mathcal{L}(\mathbb{V}, \mathbb{W})$. Prove there is a basis $(v_1, \ldots, v_M)$ for $\mathbb{V}$. and a basis $(w_1, \ldots, w_N)$ of $\mathbb{W}$ such that all the entries of $M(T)$ are 0 except for $1's$ in the (j, j) entries, which are 1 for $j = 1, \ldots, dim(range(T))$.

Let $J = dim(range(T))$. Start with a basis $(u_{J+1}, \ldots, u_M)$ for the null space of T. Extend this to a basis $(v_1, \ldots, v_J, u_{J+1}, \ldots, u_M)$ for $\mathbb{V}$. Let $w_j = Tv_j$ for $j = 1, \ldots, J$. Since T is injective on $span(v_1, \ldots, v_J)$, the vectors w_j are linearly independent. Now extend this list to a basis for $\mathbb{W}$.

3.C # 6 Suppose $\mathbb{V}$ and $\mathbb{W}$ are finite dimensional and $T \in \mathcal{L}(\mathbb{V}, \mathbb{W})$. Prove that $dim(range(T)) = 1$ if and only if there are bases for $\mathbb{V}$ and $\mathbb{W}$ such that all entries of $M(T)$ are 1.

Suppose the bases for $\mathbb{V}$ and $\mathbb{W}$ are $v_1, \ldots, v_N$ and $w_1, \ldots, w_M$ respectively. Assume first that all entries of $M(T)$ are 1. Then for each basis vector v_n we have

$$Tv_n = w_1 + \cdots + w_M,$$

and the range of T, which is the span of the T_n is one dimensional.

Now assume $dim(range(T)) = 1$. First pick a basis $(v_2, \dots, v_N)$ for the null space of T, and extend it to a basis $(v_1, v_2, \dots, v_N)$ for $\mathbb{V}$. Make a second basis for $\mathbb{V}$ with vectors $u_1 = v_1$, $u_n = v_n + v_1$ for $n > 1$. Then $w = Tu_1 \ne 0$, and $Tu_n = w$ for each $n = 1, \dots, N$.

Let $w_1 = w$ and extend the list w_1 to a basis $(w_1, \dots, w_M)$ for $\mathbb{W}$. Now define a new basis with $x_m = w_m$ for $m \ge 2$ and $x_1 = w_1 - w_2 - \cdots - w_M$. Then

$$w = x_1 + x_2 + \dots x_M,$$

and the matrix $M(T)$ has all entries equal to 1.

3.C # 12 Find 2×2 matrices A and C such that $AC \ne CA$.

Take

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Then

$$AC = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}, \quad CA = \begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix}.$$

3.D # 1 Suppose $T \in \mathcal{L}(\mathbb{U}, \mathbb{V})$ and $S \in \mathcal{L}(\mathbb{V}, \mathbb{W})$ are both invertible linear maps. Prove that $ST \in \mathcal{L}(\mathbb{U}, \mathbb{W})$ is invertible and $(ST)^{-1} = T^{-1}S^{-1}$.

Notice that $(ST)(T^{-1}S^{-1}) = I$ and $(T^{-1}S^{-1})(ST) = I$. By definition ST is invertible.

3.D # 7 Suppose $dim(\mathbb{V}) = N$ and $dim(\mathbb{W}) = M$. Let $v \in \mathbb{V}$. Let

$$E = \{T \in \mathcal{L}(\mathbb{V}, \mathbb{W}) : Tv = 0\}.$$

Show that E is a subspace of $\mathcal{L}(\mathbb{V}, \mathbb{W})$ and find $dim(E)$ if $v \ne 0$.

If $T_1 v = 0$ and $T_2 v = 0$ then $(T_1 + T_2)v = 0$, and similarly for $\alpha T_1 v$, so E is a subspace.

If $v \ne 0$, let $v_1 = v$ and extend the list (v_1) to a basis $(v_1, \dots, v_N)$ for $\mathbb{V}$. Given $T \in E$, define a linear map T_1 from $span(v_2, \dots, v_N)$ to $\mathbb{W}$ by

$$T_1 v_n = T v_n, \quad n \ge 2.$$

This mapping is linear from E to $\mathcal{L}(span(v_2, \dots, v_N), \mathbb{W})$, and has an inverse taking T_1 to T where

$$T v_n = T_1 v_n, \quad n \ge 2, \quad T v_1 = 0.$$

Thus E and $\mathcal{L}(span(v_2,\dots,v_N),\mathbb{W})$ have the same dimension, $(N-1)\times M$.

3.D # 9 Suppose that $\mathbb{V}$ is finite dimensional and $S,T\in\mathcal{L}(\mathbb{V})$. Prove that ST is invertible if and only if both S and T are invertible.

We use Proposition 3.17, which says S is invertible if and only if S is both injective and surjective.

Suppose S and T are invertible. Then S and T are both injective and surjective, so ST is injective (problem 3.6) and ST is surjective. Thus ST is invertible

Suppose ST is invertible. Since $\text{null}T\subset\text{null}ST=\{0\}$, T is injective. By Theorem 3.21, T is invertible.

Also, since ST is invertible, it is surjective. That is, for any $w\in\mathbb{V}$ there is a $v\in\mathbb{V}$ such that

$$(ST)v=w.$$

Rewriting this as $S(Tv)=w$ we see that S is surjective. By Theorem 3.21, S is invertible.

3.D # 10 Suppose that $\mathbb{V}$ is finite dimensional and $S,T\in\mathcal{L}(\mathbb{V})$. Prove that $ST=I$ if and only if $TS=I$.

Suppose that $ST=I$. The identity map I is invertible, so by problem 3.22 both S and T are invertible. Multiply $ST=I$ on the right by T^{-1} to get $S=T^{-1}$. We then have

$$TT^{-1}=TS=I.$$

Of course the implication $TS=I$ implies $ST=I$ follows by reversing the roles of S and T.