Math 4130/5130 Homework 7

4. # 4 Suppose m and n are positive integers with $m \leq n$. Prove that there exists a polynomial $p \in \mathcal{P}_n(\mathbb{F})$ with exactly m distinct roots.

Since $\mathcal{P}_n(\mathbb{F})$ is the vector space of polynomials of degree at most n, one can simply take

$$p(z) = (z - z_1) \cdots (z - z_m)$$

with distinct roots $z_1, \ldots, z_m$. If you want the degree to be exactly n, then

$$p(z) = (z - z_1) \cdots (z - z_{m-1})(z - z_m)^{n-m+1}$$

will do the trick.

4 # 6 Suppose $p \in \mathcal{P}(\mathbb{F})$ has degree m. Prove that p has m distinct roots if and only if p and its derivative p' have no roots in common.

Whether or not the roots are distinct, we can factor $p(z)$ as

$$p(z) = c(z - z_1) \cdots (z - z_m), \quad c \neq 0,$$

and by the product rule

$$p'(z) = c \sum_{n=1}^{m} \prod_{k \neq n} (z - z_k).$$

If a root of p, say z_1, is repeated, then the formula shows that $p'(z_1) = 0$. If no root is repeated then

$$p'(z_j) = c \prod_{k \neq j} (z_j - z_k) \neq 0, \quad j = 1, \ldots, m.$$

4 # 10 Suppose m is a nonnegative integer and $p \in \mathcal{P}_m$ is a polynomial of degree at most m such that there are $m+1$ distinct real numbers $x_0, \ldots, x_m$ with $p(x_j) \in \mathbb{R}$. Prove that the coefficients of p are real.

Here are two proofs.

1. Consider the linear map from the real vector space $\mathcal{P}_{m,\mathbb{R}}$ of polynomials of degree at most m with real coefficients to the real valued functions on the set $x_0, \ldots, x_m$. Both spaces have dimension $m + 1$. The null space is the set of polynomials of degree at most m vanishing at $m + 1$ points. This must be the zero polynomial, so the null space is $\{0\}$ and the map is surjective. Thus there is at least one $p \in \mathcal{P}_{m,\mathbb{R}}$ with the given values.

Now suppose $q(z)$ has complex coefficients, degree at most m, and $q(x_j) = p(x_j)$ for $j = 0, \ldots, m$. Then $q(z) - p(z)$ vanishes at $m+1$ points, so $q(z) - p(z) = 0$. Thus $p(z)$ is unique.

2. The second proof is more constructive, using Lagrange interpolating polynomials. Define

$$p_k(x) = \frac{\prod_{j=0}^{m}(x - x_j)}{x - x_k},$$

where the factors $(x - x_k)$ cancel and leave a polynomial of degree m which vanishes at $x_j \ne x_k$ and does not vanish at x_k. Suppose $p(x_j) = y_j$. We can write $p(x)$ as

$$p(x) = \sum_{k=0}^{m} y_k \frac{p_k(x)}{p_k(x_k)}.$$

This polynomial has real coefficients, and is unique among all polynomials with complex coefficients as in proof 1.

5.A # 2 Suppose $S, T \in \mathcal{L}(\mathbb{V})$ are such that $ST = TS$. Prove that null(S) is invariant under T.

If $Sv = 0$ then $S(Tv) = T(Sv) = T(0) = 0$.

5.A # 4 Suppose $T \in \mathcal{L}(\mathbb{V})$. Prove that if $U_1, \ldots, U_M$ are subspaces of $\mathbb{V}$ invariant under T, then $U_1 + \cdots + U_M$ is invariant under T.

Suppose $v = u_1 + \cdots + u_N \in U_1 + \cdots + U_M$, with $u_n \in U_n$. Then

$$Tv = Tu_1 + \cdots + Tu_N.$$

Since each U_n is invariant under T, $Tu_n =\in U_n$, so $Tv \in U_1 + \cdots + U_M$.

5.A # 5 Suppose $T \in \mathcal{L}(\mathbb{V})$. Prove that the intersection of any collection of subspaces of $\mathbb{V}$ invariant under T is invariant under T.

Suppose we have a set of subspaces $\{U_r\}$, with each U_r invariant under T. Let $v \in \cap_r U_r$. Then $Tv \in U_r$ for each r, and so $\cap_r U_r$ is invariant under T.

5.A # 6 Prove or give a counterexample: if U is a subspace of $\mathbb{V}$ that is invariant under every operator on $\mathbb{V}$, then $U = \{0\}$ or $U = \mathbb{V}$.

We'll prove the contrapositive: if U is a subspace of $\mathbb{V}$ and $U \ne \{0\}$ and $U \ne \mathbb{V}$, then there is an operator T on $\mathbb{V}$ such that U is not invariant under T.

Let $(u_1, \ldots, u_M)$ be a basis for U, which we extend to a basis

$$(u_1, \ldots, u_M, v_1, \ldots, v_N)$$

of $\mathbb{V}$. The assumption $U \neq \{0\}$ and $U \neq \mathbb{V}$ means that $M \geq 1$ and $N \geq 1$. Define a linear map T by

$$Tu_1 = v_1, \quad Tu_m = 0, \quad m > 1, \quad Tv_n = 0.$$

Since $v_1 \notin U$, the subspace U is not invariant under the operator T.

5.A # 8 Define $T \in \mathcal{L}(\mathbb{F}^2)$ by

$$T(w, z) = (z, w).$$

Find all eigenvalues and eigenvectors of T.

Suppose $(w, z) \neq (0, 0)$ and

$$T(w, z) = (z, w) = \lambda(w, z).$$

Then

$$z = \lambda w, \quad w = \lambda z.$$

Of course this leads to

$$w = \lambda z = \lambda^2 w, \quad z = \lambda w = \lambda^2 z.$$

Since $w \neq 0$ or $z \neq 0$, we see that $\lambda^2 = 1$, so $\lambda = \pm 1$.

A basis of eigenvectors is

$$(w_1, z_1) = (1, 1), \quad (w_2, z_2) = (-1, 1),$$

and they have eigenvalues 1 and -1 respectively.

5.A # 9 Define $T \in \mathcal{L}(\mathbb{F}^3)$ by

$$T(z_1, z_2, z_3) = (2z_2, 0, 5z_3).$$

Find all eigenvalues and eigenvectors of T.

Suppose $(z_1, z_2, z_3) \neq (0, 0, 0)$ and

$$T(z_1, z_2, z_3) = (2z_2, 0, 5z_3) = \lambda(z_1, z_2, z_3).$$

If $\lambda = 0$ then $z_2 = z_3 = 0$, and one checks easily that

$$v_1 = (1, 0, 0)$$

is an eigenvector with eigenvalue 0. If $\lambda \neq 0$ then $z_2 = 0$,

$$2z_2 = \lambda z_1 = 0, \quad 5z_3 = \lambda z_3,$$

so $z_1 = 0$ and $\lambda = 5$. The eigenvector for $\lambda = 5$ is

$$v_2 = (0, 0, 1).$$

These are the only eigenvalues, and each eigenspace is one dimensional.

5.A # 19 Suppose N is a positive integer and $T \in \mathcal{L}(\mathbb{F}^N)$ is defined by

$$T(x_1, \dots, x_N) = (x_1 + \cdots + x_N, \dots, x_1 + \cdots + x_N).$$

Find all eigenvalues and eigenvectors of T.

First, any vector of the form

$$v_1 = (\alpha, \dots, \alpha), \quad \alpha \in \mathbb{F},$$

is an eigenvector with eigenvalue N. If v_2 is any vector

$$v_2 = (x_1, \dots, x_N), \quad \sum_n x_n = 0,$$

then v_2 is an eigenvector with eigenvalue 0.

Here are N independent eigenvectors:

$$v_1 = (1, 1, \dots, 1),$$

and

$$v_n = (1, 0, \dots, 0) - E_n, \quad n \geq 2,$$

where E_n denotes the n-th standard basis vector.