# Boundary Value Problems for Infinite Metric Graphs

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**Abstract**

The second derivative operator $-D^2$ with standard interior vertex conditions is considered on infinite metric graphs with finite volume or other smallness conditions. A large collection of self adjoint domains defined by local boundary conditions on the metric space completion of the graph is constructed for a rich family of metric graphs that are 'weakly connected'. The developed techniques also provide the solution of the Dirichlet problem for harmonic functions.

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# 1 Introduction

Recently, there has been a burst of interest in differential equations on metric graphs $\mathcal{G}$, that is topological graphs whose edges are line segments, typically equipped with a metric and measure extending the standard ones for line segments. To formulate well posed graph versions of problems in quantum mechanics, heat flow, or wave motion, one usually wants a self adjoint realization of the second derivative operator $-D^2$ or a Schrödinger operator $-D^2 + q$ on the function space $L^2(\mathcal{G})$. Since the operators $-D^2$ and $-D^2 + q$ will often be self adjoint on the same domains [15, p. 162], our attention is restricted to $-D^2$.

Modeling problems in biology can motivate the study of differential operators on metric graphs. Primary examples are the one space dimensional models used for computationally tractable simulations of the human circulatory system [3], which transports blood, nutrients, waste products, and heat. A high fidelity graphical representation of the circulatory system has a huge number of edges with an apparently highly irregular structure, so questions about rather general infinite graph models and limiting behavior as the number of edges grows appear almost immediately.

The main aim of this work is to establish the existence and description of multiple self adjoint extensions of the standard symmetric operator $-D^2$ on infinite graphs, particularly graphs relevant for circulatory system modeling. When multiple extensions exist, one would like to know if self adjoint domains can be described using local boundary conditions, in a manner analogous to the classical theories of Sturm-Liouville problems for an interval, or boundary value problems for partial differential equations. The related problem of approximating operators on infinite graphs by operators on finite graphs is deferred; see [18] for a promising method of analysis.

The construction of self adjoint domains for $-D^2$ will lead us to consider connections between topological conditions and function theory on the metric completions of infinite graphs. For many infinite graphs, including connected graphs with finite volume, we will demonstrate the existence of distinct self adjoint realizations of $-D^2$ associated with boundary conditions of the form

$$f(x) = 0, \quad x \in \Omega, \quad f'(y) = 0, \quad y \in \Omega^c,$$

where $\Omega$ is a subset of the appropriately defined boundary of the metric completion $\overline{\mathcal{G}}$ of $\mathcal{G}$. As a by-product of the analysis, we will also solve the

Dirichlet problem for harmonic functions with prescribed boundary values. The results are developed in three subsequent sections.

The next section discusses metric graphs and their completions. The main hypothesis for graphs in this study is that they are 'weakly connected', meaning that any two points in $\overline{\mathcal{G}}$ can be separated by deleting a finite set of points from $\mathcal{G}$. This condition generalizes a property of trees, but holds in considerable generality. In particular, all connected graphs of finite volume are shown to be weakly connected. For weakly connected graphs $\mathcal{G}$, the boundary $\partial\overline{\mathcal{G}}$ is a totally disconnected set.

The third section discusses the function theory of metric graphs. A version of the Sobolev space $H^1$ is introduced, and basic properties established. Quadratic forms, the Friedrich's extension, and lower bounds for $-D^2$ on a suitable domain are discussed. The boundaries of infinite graphs may look very different from manifolds, so one is motivated to find a class of functions on graphs which can be used in place of the smooth functions on a manifold with smooth boundary. For this purpose we introduce an algebra $\mathcal{A}$ of continuous functions on $\overline{\mathcal{G}}$ with derivative zero outside a finite set of edges. The condition that $\mathcal{A}$ separates points of $\overline{\mathcal{G}}$ is shown to be equivalent to the weak connectivity of $\overline{\mathcal{G}}$.

The fourth section considers boundary value problems for infinite graphs, beginning with harmonic functions and the Dirichlet problem. The Dirichlet problem, which asks for the existence of a harmonic extension for every continuous function on $\partial\overline{\mathcal{G}}$, is solved for a large class of weakly connected compact graph completions. We conclude by defining many distinct self adjoint operators $-D^2$ on graphs with finite volume by means of local boundary conditions.

# 2 Metric Graphs and their Completions

## 2.1 Graphs

In this work a graph $\mathcal{G}$ will have a countable vertex set $\mathcal{V}$ and a countable set $\mathcal{E}$ of edges $e_n$. The graphs will be locally finite; that is, each vertex appears in at least one, but only finitely many edges. We will not directly consider loops and multiple edges with the same vertices, although these can be incorporated by adding vertices. Each edge has a positive weight (length) $l_n$. For convenience in defining differentiation, the edges are initially assumed to be directed, but this plays no essential role.

A topological graph also denoted $\mathcal{G}$ may be constructed from this data [13, p. 190]. For each directed edge $e_n$, let $[a_n, b_n]$ be a real interval of length $l_n$, and let $\alpha_j \in \{a_n, b_n\}$. Identify those interval endpoints $\alpha_j$ whose corresponding edge endpoints are the same vertex $v$. Since every point in $\mathcal{G}$ may be covered by an open set having nonempty intersection with only finitely many edges, every compact set in $\mathcal{G}$ is contained in a finite union of closed edges $e_n$.

The Euclidean length on the intervals may be extended to paths consisting of finitely many nonoverlapping intervals by addition, and a metric $d(p_1, p_2)$ on $\mathcal{G}$ is defined as the infimum of the lengths of paths joining $p_1$ and $p_2$. A graph is locally path connected, so [8, p. 90] a connected graph is path connected. Suppose $\gamma : [a, b] \to \mathcal{G}$ is a path joining $p_1$ and $p_2$. Such paths are assumed to have finite length, and are generally assumed to be parametrized by a constant multiple of arc length. We generally assume that $\gamma^{-1}(\mathcal{V})$ is a finite set $\{t_1, \dots, t_K\}$, that $\gamma(t_k) \ne \gamma(t_{k+1})$, and that $\gamma$ is unidirectional on the edges connecting $\gamma(t_k)$ and $\gamma(t_{k+1})$. The corresponding statements hold for $\gamma : [a, t_1] \to \mathcal{G}$ and $\gamma : [t_K, b] \to \mathcal{G}$.

We distinguish interior vertices, which have more than one incident edge, from boundary vertices which have a single incident edge. Edges $e_k$ incident on a vertex $v$ are denoted $e_k \sim v$. The interior of the topological graph $\mathrm{int}(\mathcal{G})$ will be the complement of the set of boundary vertices.

Suppose $\mathcal{G}^0$ is a subgraph of $\mathcal{G}$. A new graph $\mathcal{G} \setminus \mathcal{G}^0$ may be defined by deleting $\mathcal{G}^0$ from $\mathcal{G}$. The new edgeset will be $\mathcal{E} \setminus \mathcal{E}^0$. The new vertex set will be the vertices from $\mathcal{V}$ with at least one incident edge from $\mathcal{E} \setminus \mathcal{E}^0$. The same construction may be used to delete a given subset of edges instead of a prescribed subgraph.

## 2.2 Metric Completions

As a metric space, $\mathcal{G}$ has a completion $\overline{\mathcal{G}}$. A development of the metric space theory needed for this study of $\overline{\mathcal{G}}$ may be found in [17, pp.139–170]. Lebesgue measure on intervals extends in the obvious way from the edges to $\mathcal{G}$; in particular, the volume of the graph is simply the sum of the lengths of the edges. This measure is extended to $\overline{\mathcal{G}}$ (see [17, p. 257], problem 2) by defining the measure of $\overline{\mathcal{G}} \setminus \mathcal{G}$ to be 0.

Recall that a metric space $X$ is totally bounded if for every $\epsilon > 0$ there is a finite set $x_1, \dots, x_n \in X$ such that $\cup_k B_\epsilon(x_k)$ covers $X$. A metric space is compact if and only if it is complete and totally bounded. This gives a picture of graphs with compact completion.

**Proposition 2.1.** *A graph $\mathcal{G}$ has compact completion $\overline{\mathcal{G}}$ if and only if for every $\epsilon > 0$ there is a finite set of edges $e_k$, $k = 1, \dots, n$ such that for every $y \in \mathcal{G}$ there is a edge $e_k$ and a point $x_k \in e_k$ such that $d(x_k, y) < \epsilon$.*

**Corollary 2.2.** *A connected graph $\mathcal{G}$ with finite volume has compact completion $\overline{\mathcal{G}}$.*

In addition to requiring a graph to have finite volume, or a compact metric completion, a third way of keeping control of the size an infinite graph is to require that it have finite diameter. Recall that the diameter of a graph is

$$\operatorname{diam}(\mathcal{G}) = \sup_{x,y \in \mathcal{G}} d(x,y).$$

A simple 'comb' example shows that even a tree can have finite diameter but not have a compact completion.

We will define $\partial\mathcal{G}$, the boundary of $\mathcal{G}$, to be the collection of boundary vertices of $\mathcal{G}$, and the boundary of $\overline{\mathcal{G}}$, $\partial\overline{\mathcal{G}}$, to be the complement of the interior of $\mathcal{G}$ in $\overline{\mathcal{G}}$. Since $\mathcal{G}$ is locally finite, the interior of $\mathcal{G}$ is an open set, and so $\partial\overline{\mathcal{G}}$ is closed.

**Lemma 2.3.** *Every $x \in \partial\overline{\mathcal{G}}$ is either a boundary vertex or the limit in $\overline{\mathcal{G}}$ of a sequence $\{v_n\}$ of distinct vertices from $\mathcal{G}$.*

*Proof.* By definition, $x \in \partial\overline{\mathcal{G}}$ is either a boundary vertex or the limit of a Cauchy sequence $x_n \in \mathcal{G}$ with $x \notin \mathcal{G}$. Suppose $x$ is not a boundary vertex. Pick $\epsilon > 0$ and find $M$ such that $d(x_M, x) < \epsilon/2$ and $d(x_M, x_n) < \epsilon/2$ for $n \geq M$. Let $e$ be a closed edge containing $x_M$. Since $x$ is not a boundary vertex and $x$ is not in the interior of $\mathcal{G}$, there must be a distinct point $x_N \notin e$ such that $d(x_N, x_M) < \epsilon/2$. Replacing $x_M$ by one of the vertices $v_M$ of $e$ gives $d(x_N, v_M) < \epsilon/2$, so $d(v_M, x) < \epsilon$. $\square$

For paths $\gamma : [a,b] \to \overline{\mathcal{G}}$ with $\gamma : (a,b) \to \mathcal{G}$, we typically assume that $\gamma : [c,d] \to \mathcal{G}$ has the path properties discussed above if $a < c < d < b$.

**Proposition 2.4.** *Suppose $\mathcal{G}$ is connected. Then for any two points $x, y \in \overline{\mathcal{G}}$, there is a path $\gamma : [0,1] \to \overline{\mathcal{G}}$ such that $\gamma(0) = x$, $\gamma(1) = y$, and $\gamma(t) \in \mathcal{G}$ for $0 < t < 1$.*

*Proof.* The interesting case is when $x$ and $y$ are distinct points in $\overline{\mathcal{G}} \backslash \mathcal{G}$. Then there are Cauchy sequences $\{x_n\}, \{y_n\}$ such that $x_m$, $y_n$ are distinct vertices in $\mathcal{G}$, and $x = \lim_{n\to\infty} x_n$, $y = \lim_{n\to\infty} y_n$. By passing to subsequences we may assume that $\{x_n\}$ and $\{y_n\}$ are fast Cauchy sequences, satisfying

$$\sum_{n=1}^{\infty} d(x_n, x_{n+1}) < \infty, \quad \sum_{n=1}^{\infty} d(y_n, y_{n+1}) < \infty.$$

Since

$$d(x_1, y_1) = \inf_{\rho} \ \text{length}(\rho),$$

where $\rho$ is a path in $\mathcal{G}$ joining $x_1$ to $y_1$, we may start by selecting a path $\sigma$ in $\mathcal{G}$ joining $x_1$ to $y_1$ with $0 \le t \le 2d(x_1, y_1)$. Then for $N \ge 2$ inductively extend $\sigma(t)$ with a new segment $\sigma : [t_{N-1}, t_N] \to \mathcal{G}$ joining $x_{N-1}$ to $x_N$, with $\sigma(t_{N-1}) = x_{N-1}$, $\sigma(t_N) = x_N$, and

$$\text{length}(\sigma(t)) \le 2d(x_{N-1}, x_N), \quad t_{N-1} \le t \le t_N.$$

Similarly, extend $\sigma$ in the opposite direction, $\sigma : [\tau_N, \tau_{N-1}] \to \mathcal{G}$, with $\sigma(\tau_{N-1}) = y_{N-1}$, $\sigma(\tau_N) = y_N$, and

$$\text{length}(\sigma(t)) \le 2d(y_{N-1}, y_N), \quad \tau_N \le t \le \tau_{N-1}.$$

Since $\{x_n\}$ and $\{y_n\}$ are fast Cauchy sequences, the inductively defined $\sigma(t) : (a, b) \to \mathcal{G}$ is continuous on a bounded interval $(a, b)$. Since the length of a path satisfies

$$\text{length}(\sigma) = \sup \sum d(\sigma(t_k), \sigma(t_{k+1})),$$

the supremum taken over finite partitions of $(a, b)$, we have

$$d(x_n, \sigma(t)) + d(\sigma(t), x_{n+1}) \le 2d(x_n, x_{n+1}), \quad t_n \le t \le t_{n+1},$$

which implies that $\sigma$ extends continuously to $[a, b]$ with $\sigma(a) = y$ and $\sigma(b) = x$. Now reparametrize $\sigma$ to get the desired path $\gamma$.

□

It will be useful to distinguish a class of metric graphs. Say that $\mathcal{G}$ (or $\overline{\mathcal{G}}$) is *weakly connected* if for every pair of distinct points $x, y \in \overline{\mathcal{G}}$, there is a finite set of points $W = \{w_1, \ldots, w_K\} \subset \text{int}(\mathcal{G})$ separating $x$ from $y$. That is, there are disjoint open sets $U, V$ with $\overline{\mathcal{G}} \setminus W = U \cup V$, and with $x \in U$, $y \in V$. We emphasize that $W$ should come from the interior of the underlying graph, not the completion.

Of course a finite graph will be weakly connected. To construct a graph without this property (see Figure A), consider the planar vertex set

$$\mathcal{V} = \{v_n = (1/n, 1), w_n = (1/n, -1), n = 1, 2, 3, \dots\}.$$

The edges are $(v_n, w_n)$, $(v_n, v_{n+1})$, $(w_n, w_{n+1})$. Edge lengths are induced from the Euclidean metric. In this planar representation, the completion $\overline{\mathcal{G}}$ is obtained by adding the points $x = (0, 1)$ and $y = (0, -1)$. The points $x$ and $y$ may not be separated by deleting any finite set of points from $\mathcal{G}$. Notice that if the vertex set is changed to

$$\mathcal{V}' = \{v_n = (1/n, 1/n), w_n = (1/n, -1/n), n = 1, 2, 3, \dots\},$$

the analogous construction gives a topologically equivalent graph whose completion is obtained by adding the single point $(0, 0)$. This graph is weakly connected.

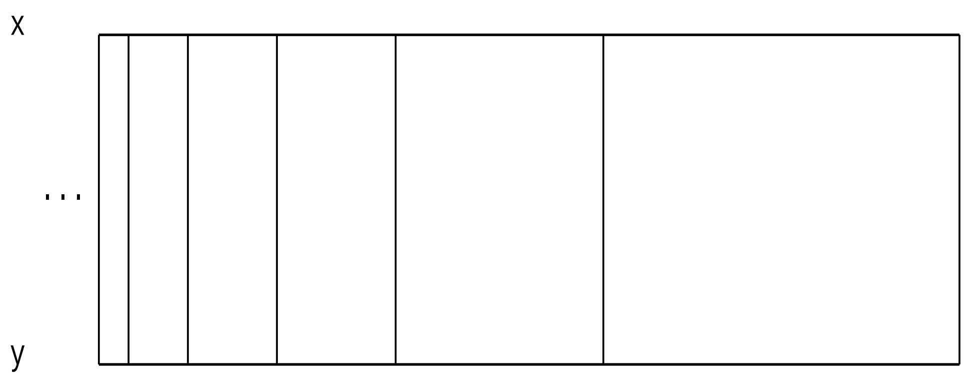


Figure A: A graph which is not weakly connected

**Proposition 2.5.** *If $\mathcal{T}$ is a tree, then $\overline{\mathcal{T}}$ is weakly connected.*

*Proof.* Suppose $x$ and $y$ are distinct points in $\overline{\mathcal{T}}$. By Lemma 2.4 there is a path $\gamma : [0, 1] \to \overline{\mathcal{T}}$ joining $x$ and $y$ with $\gamma(t) \in \mathcal{T}$ for $0 < t < 1$. Pick $z \in \mathcal{T}$ in the interior of an edge on the path $\gamma$ which is distinct from $x$ and $y$.

Because $\mathcal{T}$ is a tree there is a well defined continuous function $s$ (signed distance from $z$) such that $s(z) = 0$, $s(x) > 0$, $s(y) < 0$, and $|s(w)| = d(w, z)$.

The function $s$ is uniformly continuous on $\mathcal{T}$, so it extends continuously to $\overline{\mathcal{T}}$ [17, p. 149]. The desired separation of $\overline{\mathcal{T}}$ is provided by the sets

$$U = \{s(w) > 0\}, \quad V = \{s(w) < 0\}.$$

$\square$

Notice too that adding finitely many edges to a graph does not change its being weakly connected. Recall that a topological space is totally disconnected if its connected components are points.

**Proposition 2.6.** *If $\mathcal{G}$ is a weakly connected graph, then $\partial\overline{\mathcal{G}}$ is totally disconnected.*

*Proof.* Suppose that $x$ and $y$ are distinct points in $\partial\overline{\mathcal{G}}$. Let $U, V$ be a separation of $\overline{\mathcal{G}} \setminus W$ as described in the definition of a weakly connected graph. Then

$$U \cap \partial\overline{\mathcal{G}} = V^c \cap \partial\overline{\mathcal{G}}$$

is both closed and open in $\partial\overline{\mathcal{G}}$, so that $x$ and $y$ lie in different connected components.

$\square$

Call a set clopen if it is both closed and open. Totally disconnected compact metric spaces have a rich collection of clopen sets.

**Proposition 2.7.** *Suppose $\Omega$ is a totally disconnected compact metric space. For any $x \in \Omega$ and $\epsilon > 0$, there is a clopen set $U$ such that*

$$x \in U \subset B_\epsilon(x).$$

*Proof.* The set

$$K = \{y \in \Omega \mid d(x,y) \geq \epsilon\}$$

is compact. For each $y \in K$ there is a clopen set $V_y$ such that $y \in V_y$ and $x \notin V_y$. The collection of all such $V_y$ is an open cover of $K$, so there is a finite subcover $V_1, \dots, V_N$. Define $U$ to be the complement of $\cup_{n=1}^N V_n$ . $\square$

The next results prove that a connected metric graph with finite volume is weakly connected. The first step is to understand the structure of $\partial\overline{\mathcal{G}}$ as viewed from an arbitrary $x \in \overline{\mathcal{G}}$. Assume that $\mathcal{G}$ is connected and has finite volume. By Corollary 2.2 $\overline{\mathcal{G}}$ is compact. For $x \in \overline{\mathcal{G}}$, define the set $RP(x)$ (the radial projection for $x$) to be the range of the function $d(x,\cdot) : \partial\overline{\mathcal{G}} \to \mathbb{R}$, so that $RP(x)$ is the set of numbers $r \geq 0$ such that there is some $y \in \partial\overline{\mathcal{G}}$ with $dist(x,y) = r$.

**Lemma 2.8.** *If $\mathcal{G}$ is connected and has finite volume, then the set $RP(x)$ is a compact set of Lebesgue measure* $0$.

*Proof.* Since $\partial\overline{\mathcal{G}}$ is closed in the compact set $\overline{\mathcal{G}}$, and the function $d(x,\cdot) : \partial\overline{\mathcal{G}} \to \mathbb{R}$ is continuous, the set $RP(x)$ is compact.

Pick $\epsilon > 0$. We may delete a finite set of edges from $\mathcal{G}$, so that the remaining graph $\mathcal{G}_\epsilon$ has finitely many connected components and total volume less than $\epsilon$. By Lemma 2.3, the boundary $\partial\overline{\mathcal{G}}_\epsilon$ differs from $\partial\overline{\mathcal{G}}$ by a finite set of points.

Consider the image of $d(x,\cdot) : \mathcal{G}_\epsilon \to \mathbb{R}$. Since $\mathcal{G}_\epsilon$ has finitely many connected components, the image is the union of finitely many intervals, and except for finitely many points the closure of the image contains $RP(x)$. By the triangle inequality, $|d(x,y) - d(x,z)| \le d(y,z)$, so the function $d(x,\cdot) : \mathcal{G}_\epsilon \to \mathbb{R}$ is distance reducing. Consequently, the set $RP(x)$ is contained in a finite union of closed intervals together with a finite set of points, with total length bounded by $\epsilon$.

$\square$

**Lemma 2.9.** *Suppose $x \in \overline{\mathcal{G}}$ and $0 < r < \infty$. For any edge $e \in \mathcal{G}$ there are at most $2$ points $y \in e$ with $d(x,y) = r$.*

*Proof.* If $v$ and $w$ are the vertices of $e$, then

$$d(x,y) = \min(d(x,v) + d(v,y), d(x,w) + d(w,y)),$$

and the functions $d(v,y)$ and $d(w,y)$ are strictly monotonic on $e$.

$\square$

**Theorem 2.10.** *If the connected metric graph $\mathcal{G}$ satisfies* $\mathrm{vol}(\mathcal{G}) < \infty$*, then for any $x \in \overline{\mathcal{G}}$ and any $\epsilon > 0$ there is a finite set $W = \{w_1, \dots, w_K\} \subset \mathcal{G}$ and an $r$ with $0 < r < \epsilon$ such that the removal of $W$ separates $x$ from the complement of the closed ball $\overline{B_r(x)}$.*

*In particular, $\mathcal{G}$ is weakly connected.*

*Proof.* By Lemma 2.8, for any $\epsilon > 0$ there are numbers $0 < R_1 < R_2 < \epsilon$ such that no points of $w \in \partial\overline{\mathcal{G}}$ satisfy $R_1 \le \mathrm{dist}(x,w) \le R_2$.

Given $r$ with $R_1 < r < R_2$, suppose there are infinitely many distinct points $w_n \in \mathcal{G}$ with $d(x,w_n) = r$. By compactness there is such a sequence which converges to a point $w$, with $d(x,w) = r$. Since each edge $e \in \mathcal{G}$ contains at most 2 points $w_n$, but every neighborhood of $w$ contains infinitely many distinct points $w_n$, every neighborhood of $w$ intersects infinitely many distinct edges, so $w \in \partial\overline{\mathcal{G}}$. But this contradicts the selection of $r$.

Thus the set $W = \{w_k \mid d(x, w_k) = r\} \subset \mathcal{G}$ is finite, and the open sets $B_r(x)$ and the complement of $\overline{B_r(x)}$ provide a separation of $\overline{\mathcal{G}} \setminus W$.

□

# 3 Function theory

## 3.1 Function spaces

The identification of edges $e_j$ with intervals facilitates the discussion of function spaces and differential operators. Let $L^2(\mathcal{G})$ denote the Hilbert space $\oplus_j L^2(e_j)$ with the inner product

$$\langle f, g\rangle_0 = \int_{\mathcal{G}} f\overline{g} = \sum_j \int_{a_j}^{b_j} f_j(x)\overline{g_j(x)}\ dx, \quad f = (f_1, f_2, \dots).$$

A related inner product for differentiable functions is

$$\langle f, g\rangle_1 = \int_{\mathcal{G}} f\overline{g} + f'\overline{g'} = \sum_j \int_{a_j}^{b_j} f_j(x)\overline{g_j(x)} + f_j'(x)\overline{g_j'(x)}\ dx,$$

with the induced quadratic form $Q_1(f) = \langle f, f\rangle_1$. Denote by $L_1^2(\mathcal{G})$ the Hilbert space consisting functions $f \in L^2(\mathcal{G})$ which are absolutely continuous on each edge, whose derivative (computed edgewise) is square integrable, with the inner product given by $\langle f, g\rangle_1$. Denote by $H^1(\mathcal{G})$ the set of functions in $L_1^2(\mathcal{G})$ which are continuous at the vertices.

Suppose that $e_1$ and $e_2$ are two edges with a common vertex $v$. Choose local coordinates representing $e_1$ as $[a, b]$ and $e_2$ as $[b, c]$, with $b$ the local coordinate value for $v$ on both edges. If $f \in H^1(\mathcal{G})$ then the Fundamental Theorem of Calculus applies on each edge [17, p. 110]. Since $f$ is continuous at $b$,

$$\int_a^c f'(t)\ dt = [f(b^-) - f(a^+)] + [f(c^-) - f(b^+)] = f(c) - f(a),$$

and the Fundamental Theorem of Calculus applies on paths in $\mathcal{G}$.

Let's note that $H^1$ is a closed subspace of $L_1^2(\mathcal{G})$. Suppose $f \in L_1^2(\mathcal{G})$ and $e$ is a closed edge of length $l$. If

$$\|f\|_1^2 = \int_{\mathcal{G}} |f|^2 + |f'|^2,$$

then there is a point $x_0 \in e$ such that

$$|f(x_0)| \le l^{-1/2} \|f\|_1. \tag{3.1}$$

For $x \in e$ the Fundamental Theorem of Calculus gives

$$f(x) = f(x_0) + \int_{x_0}^{x} f'(t)\ dt,$$

and the Cauchy-Schwarz inequality implies

$$|f(x)| \le l^{-1/2} \|f\|_1 + l^{1/2} \|f\|_1. \tag{3.2}$$

This well know argument shows that evaluation at any point on an edge (closed interval) is a continuous linear functional on $L_1^2$. Since $H^1$ is the intersection of the null space of the functionals

$$\lim_{x \in e_i \to v} f(x) - \lim_{x \in e_j \to v} f(x), \quad v \in \mathcal{V},$$

it is a closed subspace.

**Lemma 3.1.** *If $\mathcal{G}$ is a connected graph with finite diameter $R$, then there is a constant $C$ such that*

$$\sup_{x \in \mathcal{G}} |f(x)| \le C \|f\|_1, \quad f \in H^1(\mathcal{G}) \tag{3.3}$$

*In particular, a Cauchy sequence in $H^1$ is a uniformly Cauchy sequence. The functions $f$ in the unit ball $B$ of $H^1$ are uniformly equicontinuous [14, p. 29] on $\mathcal{G}$.*

*Proof.* Given any two points $x_1, x_2 \in \mathcal{G}$, let $\gamma$ be a (simple) path from $x_1$ to $x_2$ of length not greater than $2R$. Computing derivatives with respect to the orientation of edges provided by $\gamma$, we have

$$f(x_2) - f(x_1) = \int_{\gamma} f'(t)\ dt. \tag{3.4}$$

Pick an edge $e \in \mathcal{G}$ and let $x_0$ satisfy (3.1). Similar to (3.2), (3.4) gives

$$|f(x)| \le l^{-1/2}\|f\|_1 + (2R)^{1/2}\|f\|_1,$$

establishing (3.3).

Starting with (3.4), the Cauchy-Schwarz inequality leads to the (Lipshitz) estimate

$$|f(x_2) - f(x_1)| \le \int_\gamma |f'(t)| \ dt \tag{3.5}$$

$$\le \Bigl( \int_\gamma 1^2 \ dt \Bigr)^{1/2} \Bigl( \int_\gamma (f')^2 \ dt \Bigr)^{1/2} \le (2R)^{1/2}\|f\|_1,$$

establishing the desired uniform equicontinuity.

□

If $\Omega \subset \overline{\mathcal{G}}$, let $C(\Omega)$ denote the Banach space of bounded complex valued continuous functions on $\Omega$ with the usual norm $\|f\|_\infty = \sup_{x\in\Omega} |f(x)|$. The previous lemma provides useful information about extending functions $f \in H^1(\mathcal{G})$ to $C(\overline{\mathcal{G}})$.

**Lemma 3.2.** *Suppose that $\mathcal{G}$ is a connected graph with finite diameter. Then each function $f$ in $H^1$ has a unique continuous extension to $\overline{\mathcal{G}}$. If $\Omega \subset \overline{\mathcal{G}}$ is compact, then every (extended) sequence $f_n \in H^1(\mathcal{G})$ satisfying $\|f\|_1 \le 1$ has a subsequence converging uniformly in $C(\Omega)$.*

*Proof.* By [17, p. 149] the functions $f$ extend by continuity to a uniformly equicontinuous family on $\overline{\mathcal{G}}$. The Arzela-Ascoli theorem [17, p. 169] completes the result. □

The next results relate the function theory and connectivity properties of $\overline{\mathcal{G}}$. Let $\mathcal{A}$ denote the real vector space of continuous functions $\phi : \overline{\mathcal{G}} \to \mathbb{R}$ which are infinitely differentiable on the open edges of $\mathcal{G}$, with $\phi'$ satisfying the following compact support condition: there is a finite set $\{e_1, \dots, e_K\}$ of edges in $\mathcal{G}$, and a compact subset $\Omega_k$ in the interior of each $e_k$, such that $\phi'(x) = 0$ for $x \in \mathcal{G} \setminus [\cup_{k=1}^K \Omega_k]$. With pointwise multiplication the vector space $\mathcal{A}$ is an algebra which contains the constant functions.

**Lemma 3.3.** *Suppose $\mathcal{G}$ is weakly connected. If $x$ and $y$ are distinct points of $\overline{\mathcal{G}}$, there is a function $\phi \in \mathcal{A}$ such that $0 \le \phi \le 1$, $\phi(z) = 0$ for $z$ in an open neighborhood $U_1$ of $x$, and $\phi(z) = 1$ in an open neighborhood $V_1$ of $y$.*

*Proof.* The other cases being trivial, assume $x, y \in \overline{\mathcal{G}} \setminus \mathcal{G}$.

Let $W = \{w_1, \dots, w_K\}$ be a finite set of points whose removal provides a separation $\overline{\mathcal{G}} \setminus W = U \cup V$ with $x \in U$, $y \in V$, and $U \cap V = \emptyset$.

Suppose that $r > 0$ and $r$ is sufficiently small. Let $B_r = \{z \in \mathcal{G}, d(z, W) < r\}$. We may write the set $B_r$ as the pairwise disjoint union of two types of sets $I_k$. If $w_k$ is an interior point of an edge $e_k$, then $I_k$ will be an interval of radius $r$ centered at $w_k$ inside $e_k$. If $w_k$ is a vertex, then $I_k$ will be the union of a finite collection of intervals of length $r$ which are subsets of edges incident on $w_k$.

Define $U_1 = U \setminus B_r$ and $V_1 = V \setminus B_r$. Define a function $\phi(z)$ such that $\phi(z) = 0$ for $z \in U_1$ and $\phi(z) = 1$ for $z \in V_1$. It is straightforward to extend $\phi$ on each subset $I_k$ of $B_r$ so that the resulting function $\phi$ is in $\mathcal{A}$, giving the result.

$\square$

**Theorem 3.4.** *The algebra $\mathcal{A}$ separates points of $\overline{\mathcal{G}}$ if and only if $\overline{\mathcal{G}}$ is weakly connected.*

*Proof.* If $\overline{\mathcal{G}}$ is weakly connected, Lemma 3.3 shows that the algebra $\mathcal{A}$ separates points of $\overline{\mathcal{G}}$

Conversely, suppose $\mathcal{A}$ separates points of $\overline{\mathcal{G}}$. Assume that $x$ and $y$ are distinct points of $\overline{\mathcal{G}}$; with no loss of generality we may assume $x$ and $y$ are in $\overline{\mathcal{G}} \setminus \mathcal{G}$, and that they lie in the same path component of $\overline{\mathcal{G}}$. Let $\phi \in \mathcal{A}$, with $\phi(x) < \phi(y)$. Let $e_1, \dots, e_K$ be the finite collection of open edges with $\phi'(z) \ne 0$ for some $z \in e_k$, and let $v_k, w_k$ be the endpoints of $e_k$ with $\phi(v_k) \le \phi(w_k)$. After a modification on the edges $e_k$, $\phi(t)$ may be assumed to be nondecreasing as $t$ runs from $v_k$ to $w_k$, with $\phi'(t) > 0$ unless $\phi(t) = \phi(v_k)$ or $\phi(t) = \phi(w_k)$.

Let $F = \{\phi(v_k), \phi(w_k), k = 1, \dots, K\}$. Using Proposition 2.4 one may show that if $z \in \overline{\mathcal{G}} \setminus [\cup_k e_k]$, then $\phi(z) \in F$. Pick a value $c \in (\phi(x), \phi(y))$ such that $c \notin F$. Then the set $\phi^{-1}(c)$ is finite, and $U = \phi^{-1}(-\infty, c)$, $V = \phi^{-1}(c, \infty)$, provides the desired separation of $x$ and $y$.

$\square$

The following refinement is available.

**Lemma 3.5.** *Suppose $\overline{\mathcal{G}}$ is weakly connected. If $\Omega$ and $\Omega_1$ are disjoint compact subsets of $\overline{\mathcal{G}}$, then there is a function $f \in \mathcal{A}$ such that $0 \le f \le 1$,*

$$f(x) = 1, \quad x \in \Omega, \quad f(y) = 0, \quad y \in \Omega_1.$$

*Proof.* Fix $y \in \Omega_1$, and assume that $x \in \Omega$. By Lemma 3.3 there is a function $f_x \in \mathcal{A}$ such that $0 \le f_x \le 1$, $f_x(z) = 0$ for $z$ in an open neighborhood $U_x$ of $x$, and $f_x(z) = 1$ in an open neighborhood of $y$. Cover $\Omega$ with a collection of the sets $U_x$, and let $U_1, \dots, U_N$ be a finite subcover, with corresponding functions $f_1, \dots, f_N$. The product $F_y = f_1 \cdots f_N$ will vanish on $\Omega$, but $F_y(z) = 1$ for $z$ in an open neighborhood $V_y$ of $y$.

Now cover $\Omega_1$ with a collection of the sets $V_y$, and let $V_1, \dots, V_M$ be a finite subcover, with corresponding functions $F_1, \dots, F_M$. The function

$$f = (1 - F_1) \cdots (1 - F_M)$$

has the desired properties. $\square$

For $\Omega \subset \overline{\mathcal{G}}$ closed, let $\mathcal{A}_\Omega$ denote the subalgebra of $\mathcal{A}$ vanishing on $\Omega$,

$$\mathcal{A}_\Omega = \{f \in \mathcal{A} | f(x) = 0 \text{ for all } x \in \Omega\}.$$

**Theorem 3.6.** *Suppose $\overline{\mathcal{G}}$ is weakly connected. If $\Omega$ and $\Omega_1$ are disjoint compact subsets of $\overline{\mathcal{G}}$, then the algebra $\mathcal{A}_\Omega$, restricted to $\Omega_1$, is uniformly dense in $C(\Omega_1, \mathbb{R})$.*

*Proof.* For each pair of distinct points $x$ and $y$ in $\Omega_1$ pick numbers $r$ and $R$ with $0 < r < R < d(x, y)$. The sets

$$V_1 = \{z \in \Omega_1 \mid d(z, x) \le r\}, \quad V_2 = \{z \in \Omega_1 \cup \Omega \mid d(z, x) \ge R\},$$

are disjoint, compact, and contain $x$ and $y$ respectively. By Lemma 3.5 the algebra $\mathcal{A}_\Omega$ separates points of $\Omega_1$ and vanishes nowhere on $\Omega_1$, so the result follows from the Stone-Weierstrass theorem [17, p. 212]. $\square$

In many cases the ideas in Lemma 3.5 and Theorem 3.6 may be developed further. Given a compact set $\Omega \in \overline{\mathcal{G}}$, define

$$N_R(\Omega) = \{z \in \overline{\mathcal{G}} \mid d(z, \Omega) \le R\}.$$

Suppose that $N_R(\Omega)$ is compact. Pick a number $r$ with $0 < r < R$, and define the set

$$V = \{z \in \overline{\mathcal{G}} \mid r \le d(z, \Omega) \le R\},$$

which is compact and disjoint from $\Omega$. Given a function $f \in \mathcal{A}_V$, there is a function $f_1 \in \mathcal{A}$ which agrees with $f$ on $N_R(\Omega)$ and is zero on $\overline{\mathcal{G}} \setminus N_R(\Omega)$. This observation is summarized in the next proposition.

**Proposition 3.7.** *Suppose $\Omega \subset \overline{\mathcal{G}}$ is compact, and $N_R(\Omega)$ is compact for some $R > 0$. If $0 < r < R$, then for every $f \in \mathcal{A}_V$ there is a function $f_1 \in \mathcal{A}$ satisfying*

$$f_1(z) = f(z), \quad z \in N_R(\Omega),$$
$$f_1(z) = 0, \quad z \in \overline{\mathcal{G}} \setminus N_R(\Omega).$$

## 3.2 Symmetric operators and quadratic forms

In this paper the differential operators on $L^2(\mathcal{G})$ will have a common dense domain $\mathcal{D}_0$. A function $f : \mathcal{G} \to \mathbb{C}$ will be in $\mathcal{D}_0$ if $f$ is infinitely differentiable on all open edges, and if there is a finite set of edges $e_n$, $n = 1, \dots, N$ such that the support of $f$ is a compact subset $K$ of the union of the interiors of the sets $e_n$, $K \subset \cup_{n=1}^N \mathrm{int}(e_n)$. We start with the symmetric operator $S_0 : L^2(\mathcal{G}) \to L^2(\mathcal{G})$ which acts by $-D^2$ on the domain $\mathcal{D}_0$. By working on one edge at a time the classical treatment of differential operators [6, p. 1294], [11, pp. 169–171] shows that $S_0^* : L^2(\mathcal{G}) \to L^2(\mathcal{G})$, the adjoint of $S_0 : L^2(\mathcal{G}) \to L^2(\mathcal{G})$, is a differential operator acting by $-D^2$. In addition we obtain the following lemma.

**Lemma 3.8.** *For $S_0 : L^2 \to L^2$ the domain $\mathcal{D}_1$ of $S_0^* = -D^2$ consists of functions $f \in L^2(\mathcal{G})$ such that $f'$ is absolutely continuous on each $e_n$, and $f'' \in L^2(\mathcal{G})$.*

Prescribing a domain on which $-D^2$ is self adjoint typically begins with conditions at the interior graph vertices. Physical reasoning often suggests that the functions in the domain should be continuous and satisfy a derivative condition at interior vertices $v$ with multiple incident edges.

$$\lim_{x \in e(i) \to v} f(x) = \lim_{x \in e(j) \to v} f(x), \quad e(i), e(j) \sim v. \tag{3.6}$$

$$\sum_{e \sim v} \partial_\nu f_e(v) = 0. \tag{3.7}$$

Here the derivative $\partial_\nu f_k(v) = f'$ is computed in local coordinates identifying $[a, b]$ with the edge $e_k$ of length $b - a$, and $a$ is the coordinate value for $v$. Because of their popularity, the vertex conditions (3.6) and (3.7) will be called the standard vertex conditions. The references [2, 10, 12] may be consulted for further discussion and alternatives.

The imposition of these standard interior vertex conditions leads us to two new domains. The first will be $\mathcal{D}_s$ consisting of all functions in $\mathcal{D}_1$ which satisfy the standard vertex conditions. The second domain $\mathcal{D}_K$ will consist of all functions in $\mathcal{D}_s$ which vanish identically in a neighborhood of all boundary vertices and in the complement of a finite set of edges. Let $S_K : L^2(\mathcal{T}) \to L^2(\mathcal{T})$ denote the operator acting by $-D^2$ with domain $\mathcal{D}_K$. A direct computation establishes the next result.

**Lemma 3.9.** *The operator $S_K$ on $\mathcal{D}_K$ is symmetric, with the associated positive form*

$$Q(f,g) = \int_{\mathcal{G}} f'\overline{g'}. \tag{3.8}$$

*$S_K$ is bounded below by 0, and $\mathcal{D}_s$ is the domain of its adjoint.*

Since $S_K$ is bounded below by 0, it has self adjoint extensions, including the Friedrich's extension which we denote $\mathcal{L}_K : L^2(\overline{\mathcal{G}}) \to L^2(\overline{\mathcal{G}})$. Recall that $\mathcal{L}_K$ has the same lower bound as $S_K$. Closely related to the domain of $\mathcal{L}_K$ is the space $H_0^1$, the closure of $\mathcal{D}_K$ in $H^1$.

**Theorem 3.10.** *Suppose $\overline{\mathcal{G}}$ is compact and connected. If $f \in H_0^1$, or if $f$ is in the domain of $\mathcal{L}_K$, then $f$ is continuous on $\overline{\mathcal{G}}$, and $f(x) = 0$ if $x \in \partial\overline{\mathcal{G}}$.*

*Proof.* A function $f$ in $H_0^1$ is the limit in $H^1$ of a sequence of functions $f_n \in \mathcal{D}_K$, with $f_n(x) = 0$ for $x \in \partial\overline{\mathcal{G}}$. By Lemma 3.1 and Lemma 3.2 the sequence $f_n$ convergences uniformly on $\overline{\mathcal{G}}$, so the limit function $f$ is continuous, and $f(x) = 0$ for $x \in \partial\overline{\mathcal{G}}$.

The domain of the quadratic form $Q(f,f)$ for $S_K$ is $\mathcal{D}_K$, and the closure $\overline{Q}$ of the form is obtained by completing the domain in the $H^1$ norm [15, p. 177]. Thus the domain of $\overline{Q}$ is $H_0^1$. The domain of the Friedrich's extension [4, p. 37], [14, p. 278] is the subset of $H^1$ in the domain of $\overline{Q}$ and in the domain of $S_K^*$. $\square$

If the volume of $\mathcal{G}$ is finite, distinct symmetric operators acting by $-D^2$ may be constructed by extending the domain $\mathcal{D}_K$. One simple example is obtained by adding the function 1, giving a symmetric extension of $S_K$ whose quadratic form is given by the formula (3.8). It will be convenient to have criteria which insure that various self adjoint extensions of $S_K$ have compact resolvent, so that the spectrum will consist of a discrete set of eigenvalues of finite multiplicity. Some results in this direction may be achieved by employing the form $Q$.

If $\mathcal{G}$ has finite volume then a uniformly convergent sequence also converges in $L^2$. Moreover the compactness result Lemma 3.2 will imply compactness of the resolvent for self adjoint extensions of $L_0$ whose associated quadratic form is $Q$ [16, p. 245].

**Proposition 3.11.** *If $\mathcal{G}$ has finite volume and $S_1$ is a symmetric extension of $S_K$ in $L^2(\mathcal{G})$ whose associated quadratic form is*

$$\langle S_1 f, f \rangle = \int_{\mathcal{G}} f' \overline{f'}, \quad f \in \text{domain}(S_1),$$

*then the Friedrich's extension of $S_1$ has compact resolvent.*

We want some geometric conditions which guarantee that the operator $S_K$, and so its Friedrich's extension, have a strictly positive lower bound. A simple test is provided in the next proposition.

**Proposition 3.12.** *Suppose $\mathcal{G}$ has finite volume, and $\partial\overline{\mathcal{G}} \neq \emptyset$. If $f$ is real valued in the domain of $S_K$, and*

$$\int_{\mathcal{G}} f^2 = 1,$$

*then*

$$\int_{\mathcal{G}} (f')^2 \geq vol(\mathcal{G})^{-2},$$

*so $S_K$ has a strictly positive lower bound.*

*Proof.* There is a point $x \in \overline{\mathcal{G}}$ with $f(x) = 0$, and a point $y \in \mathcal{G}$ such that $f^2(y) \geq vol(\mathcal{G})^{-1}$. Connect $y$ to $x$ by a simple path $\gamma$. By the Cauchy-Schwarz inequality

$$vol(\mathcal{G})^{-1} \leq f(y)^2 = [f(y) - f(x)]^2 = [\int_{\gamma} f'(t)\ dt]^2 \leq vol(\mathcal{G}) \int_{\mathcal{G}} |f'|^2\ dV.$$

$\square$

For graphs $\mathcal{G}$ with finite diameter but infinite volume, a strictly positive lower bound for the operator $S_K$ may be insured by a covering condition, which is satisfied for all finite diameter trees. We will say that a graph $\mathcal{G}$ with $\partial\overline{\mathcal{G}} \neq \emptyset$ has a good covering if there is a (finite or infinite) sequence of paths $\gamma_n : [0, b_n] \to \overline{\mathcal{G}}$ parametrized by arc length with the following properties:

(i) $\mathcal{G} \subset \bigcup_n \gamma_n([0, b_n])$,

(ii) $\gamma_n : (0, b_n) \to \mathcal{G}$, and $\gamma(b_n) \in \partial \overline{\mathcal{G}}$,

(iii) there is a constant $C_1 > 0$ such that $b_n \leq C_1 \mathrm{diam}(\mathcal{G})$,

(iv) there is a constant $C_2 > 0$ such that the set of $x \in \mathcal{G}$ lying in more than $C_2$ paths $\gamma_n$ has measure 0.

**Proposition 3.13.** *Suppose a finite diameter graph $\mathcal{G}$ has a good covering. Then for all $f$ in the domain of $S_K$,*

$$(S_K f, f) \geq C_1^{-2} C_2^{-1} \mathrm{diam}(\mathcal{G})^{-2} \|f\|^2.$$

*Proof.* Suppose $f$ is real valued in the domain of $S_K$, and

$$\int_{\mathcal{G}} f^2 = 1.$$

Let $x_n = \gamma(b_n)$, and recall that $f(x_n) = 0$ for any $f$ in the domain of $S_K$. Let $y_n$ be a maximum of $f$ on $\gamma_n$, and let $\sigma_n$ be a path from $x_n$ to $y_n$ with length$(\sigma_n) \leq b_n$.

By the Cauchy-Schwarz inequality

$$\int_{\gamma_n} f^2 \leq f^2(y_n) b_n = b_n [f(y_n) - f(x_n)]^2 = b_n [\int_{\sigma_n} f'(t)\, dt]^2 \leq b_n^2 \int_{\sigma_n} |f'(t)|^2\, dt.$$

Now sum over $n$ to get

$$1 = \int_{\mathcal{G}} f^2 \leq \sum_n \int_{\gamma_n} f^2 \leq \sum_n b_n^2 \int_{\sigma_n} |f'(t)|^2\, dt$$

$$\leq C_1^2 \mathrm{diam}(\mathcal{G})^2 \sum_n \int_{\gamma_n} |f'(t)|^2\, dt \leq C_1^2 C_2 \mathrm{diam}(\mathcal{G})^2 \int_{\mathcal{G}} |f'(t)|^2\, dt,$$

or

$$\int_{\mathcal{G}} (f')^2 \geq C_1^{-2} C_2^{-1} \mathrm{diam}(\mathcal{G})^{-2}.$$

□

Here we sketch an example of a compact $\overline{\mathcal{G}}$ with nonempty boundary, and with lower bound equal to 0 for $S_K$. No self adjoint extension of the constructed $S_K$ will have compact resolvent. Start with the half open interval $(0,1]$, and place vertices at 1, and at a sequence of points $x_k < 1$, with $x_{k+1} < x_k$ and $\lim_{k\to\infty} x_k = 0$. At each of the vertices $x_k$ attach a loop of length $l_k$, with $\lim_{k\to\infty} l_k = 0$. With these conditions, such a graph has finite diameter and compact completion. The set $\partial\overline{\mathcal{G}}$ will consist of the two points $0, 1$.

Continuing with the construction, let $\phi : \mathbb{R} \to \mathbb{R}$ be a $C^\infty$ function with support in the interval $(1,2)$, and with $\phi(x) = 1$ for $5/4 \le x \le 7/4$. Define a sequence of functions $\phi_n : (0,1] \to \mathbb{R}$ by requiring

$$\phi_n(x) = 2^{-n}\phi(2^n x), \quad 0 < x \le 1.$$

Of course the support of $\phi_n$ lies in $(2^{-n}, 2^{1-n})$, and the derivatives $\phi_n'(x)$ are uniformly bounded by a constant $C$.

Pick a collection of $8^n$ vertices $x_k$ inside the interval

$$I_n = (2^{-n}5/4, 2^{-n}7/4) \subset (2^{-n}, 2^{1-n}).$$

For $x \in I_n$ we have $\phi_n(x) = 2^{-n}$. Let the loops at $x_k$ have lengths $l_k = 2^{-n}$. The function $\phi_n$ extends to a function in the domain of $S_K$ if $\phi_n(z) = \phi_n(x_k)$ for all $z$ in the loop at $x_k$.

The function $\phi_n(x)$ has the properties

$$\int_{\mathcal{G}} |\phi_n|^2 \ge 1, \quad \int_{\mathcal{G}} |\phi_n'|^2 \le 2^{-n}C^2,$$

which shows that $S_K$ is not bounded below by any positive number. Since the functions $\phi_n$ are orthogonal, for any $\epsilon > 0$ there is an infinite dimensional linear manifold on which the quadratic form for $S_K$ is bounded by $\epsilon$. This shows that no self adjoint extension of $S_K$ can have compact resolvent.

# 4 Boundary Value Problems

## 4.1 Harmonic functions

In this section we consider harmonic functions on graphs. A harmonic function is defined to be a continuous function on $\mathcal{G}$ which is linear on each edge, i.e. $D^2 f = 0$, and which satisfies the standard vertex conditions (3.6) and (3.7) at each interior vertex.

We would like our harmonic functions to satisfy the mean value property at vertices. Assume that $v$ is an interior vertex with incident edges $e_1, \dots, e_N$ pointing away from $v$. Suppose that $f$ is a function which is continuous at $v$, and whose restriction $f_n$ to $e_n$ is linear. Then for $x_0 > 0$ sufficiently small and $0 < x_1 \le x_0$ we have

$$f_n(x) = f_n(x_1) + f_n' * (x - x_1), \quad 0 < x \le x_0.$$

In particular

$$f(v) = f_n(0) = f_n(x_1) - f_n' x_1.$$

The next lemma follows by taking the mean of both sides,

$$f(v) = \frac{1}{N} \sum_{n=1}^{N} f_n(x_1) - \frac{x_1}{N} \sum_{n=1}^{N} \partial_\nu f_n.$$

**Lemma 4.1.** *If $f$ is linear on the edges incident on $v$, and continuous at $v$, then $f(v)$ is the mean value of the edge values $f_n(x_1)$ for $0 < x_1 \le x_0$ if and only if $\sum_{n=1}^{N} \partial_\nu f_n = 0$ .*

The mean value property is extended in the next result.

**Lemma 4.2.** *Suppose $f$ is harmonic on $\mathcal{G}$, and $e_1, \dots, e_N$ are the edges incident on an interior vertex $v$. If $0 < x_n \le length(e_n)$, then $f(v)$ is a weighted average of the values $f_n(x_n)$ with weights*

$$w_n = \frac{1}{x_n} \Bigl( \sum_{k=1}^{N} \frac{1}{x_k} \Bigr)^{-1}.$$

*Proof.* The formula

$$f_n(x) = f_n(x_n) + f_n' * (x - x_n), \quad 0 < x \le length(e_n)$$

gives

$$f_n(0) = f_n(x_n) - f_n' x_n, \quad n = 1, \dots, N.$$

The given positive weights $w_n$ satisfy $\sum_{n=1}^{N} w_n = 1$, so

$$f(v) = \sum_{n=1}^{N} f_n(x_n) w_n - \Bigl( \sum_{k=1}^{N} \frac{1}{x_k} \Bigr)^{-1} \sum_{n} f_n' = \sum_{n=1}^{N} f_n(x_n) w_n.$$

□

As with classic harmonic functions, the mean value property for a harmonic function $f$ on a metric graph means that $f$ has an interior maximum or minimum on the connected graph $\mathcal{G}$ if and only if $f$ is constant. If $\overline{\mathcal{G}}$ is compact and $f$ extends continuously to $\overline{\mathcal{G}}$, then of course there is a max and min, and they occur on the boundary. Together with Theorem 3.10, this implies that if $\overline{\mathcal{G}}$ is compact and $\partial\overline{\mathcal{G}} \ne \emptyset$, then 0 cannot be an eigenvalue of $\mathcal{L}_K$, the Friedrich's extension of $S_K$.

Given a continuous function $F : \partial\overline{\mathcal{G}} \to \mathbb{R}$, the Dirichlet problem asks for a continuous extension $f : \overline{\mathcal{G}} \to \mathbb{R}$ which is harmonic on $\mathcal{G}$. Here is an example of a graph with finite diameter and noncompact boundary for which the Dirichlet problem is not always solvable.

As shown in Figure B, construct a graph $\mathcal{G}$ beginning with a sequence of vertices $v_n$ from the interval $[0,1]$ at the points $v_n = 2^{-n}$, and with edges $(v_n, v_{n+1})$, n = 0,1,2,.... To each of the vertices $v_n$ we then attach $2^n$ additional edges of length 1 whose other endpoints $u_{n,k}$ are boundary vertices. In this example, $\partial\overline{\mathcal{G}}$ consists of the boundary vertices $u_{n,k}$ and the boundary point at $v = \lim_n v_n$. All of these points are isolated in $\partial\overline{\mathcal{G}}$. Suppose the function $F$ satisfies

$$F(u_{n,k}) = 0, \quad F(v) = 1.$$

Then Lemma 4.2 shows that any harmonic extension $f$ must satisfy the contradictory requirements

$$\lim_{n\to\infty} f(v_n) = 1, \quad \lim_{n\to\infty} f(v_n) < 1.$$

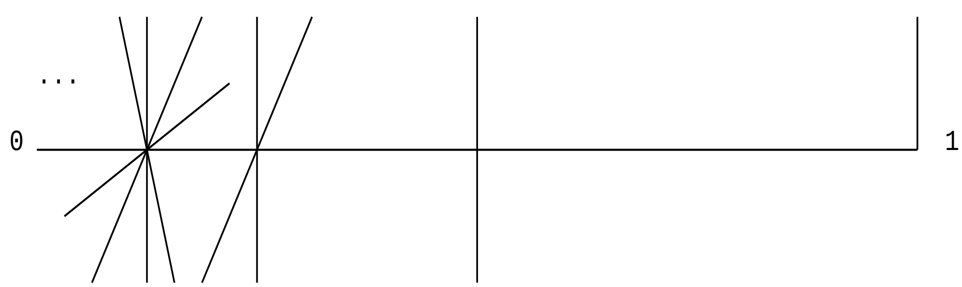


Figure B

The Dirichlet problem for $\overline{\mathcal{G}}$ can often bes solved using the algebra $\mathcal{A}$ described in Theorem 3.4.

**Theorem 4.3.** *Suppose that $\overline{\mathcal{G}}$ is weakly connected and compact. If the symmetric operator $S_K$ has a strictly positive lower bound, then every continuous function $F : \partial\overline{\mathcal{G}} \to \mathbb{C}$ has a unique extension to a continuous function $f : \overline{\mathcal{G}} \to \mathbb{C}$ that is harmonic on $\mathcal{G}$.*

*Proof.* The uniqueness is a standard consequence of the maximum principle, since the difference of any two solutions is 0 on $\partial\overline{\mathcal{G}}$.

By the Tietze extension theorem [7, p. 116], [17, p. 179] we may assume that $F$ is defined and continuous on $\overline{\mathcal{G}}$. The algebra $\mathcal{A}$ contains 1, and by Theorem 3.4 $\mathcal{A}$ separates the points of $\overline{\mathcal{G}}$. By the Stone-Weierstrass theorem [7, p. 133], for any $n > 0$ there is an element $g_n \in \mathcal{A}$ with

$$\max_{x \in \overline{\mathcal{G}}} |F(x) - g_n(x)| \le 1/n.$$

Since $g_n \in \mathcal{A}$, the identity $D^2 g_n = 0$ holds except on finitely many edges, and $D^2 g_n \in L^2(\overline{\mathcal{G}})$. Using the Friedrich's extension $\mathcal{L}_K$ and Lemma 3.10 we may solve the equation

$$-D^2 h_n = D^2 g_n, \quad h_n\big|_{\partial\overline{\mathcal{G}}} = 0.$$

The function $f_n = h_n + g_n$ is then harmonic on $\mathcal{G}$, continuous on $\overline{\mathcal{G}}$, and satisfies

$$\max_{x \in \partial\overline{\mathcal{G}}} |F(x) - f_n(x)| \le 1/n.$$

Since $\{f_n\}$ is a uniformly Cauchy sequence on $\partial\overline{\mathcal{G}}$, the maximum principle shows that it is a uniformly Cauchy sequence on $\overline{\mathcal{G}}$. Thus there is continuous limit $f : \overline{\mathcal{G}} \to \mathbb{R}$. On each edge we may pick a coordinate and write

$$(f_n - f_m)(x) - (f_n - f_m)(x_0) = \int_{x_0}^{x} (f_n' - f_m')(t)\ dt = (c_n - c_m)(x - x_0).$$

This immediately implies that $f$ is harmonic.

□

If one simply wishes to establish the existence of many harmonic functions, Corollary 3.7 may be used instead of global compactness.

**Theorem 4.4.** *Suppose that $\overline{\mathcal{G}}$ is connected, weakly connected, and has finite diameter. Assume that $\Omega \subset \partial\overline{\mathcal{G}}$ is compact, and $N_R(\Omega)$ is compact for some $R > 0$. If the symmetric operator $S_K$ has a strictly positive lower bound,* *then there is a dense subset $C_1 \subset C(\Omega, \mathbb{R})$ such that every function $F \in C_1$ has an extension to a continuous function $f : \overline{\mathcal{G}} \to \mathbb{C}$ that is harmonic on $\mathcal{G}$.*

*If $N_R(\Omega)$ has finite volume, the harmonic function $f$ may be taken to be square integrable on $\overline{\mathcal{G}}$.*

*Proof.* Suppose $G : \Omega \to \mathbb{R}$ is continuous. For $0 < r < R$, define

$$V = \{z \in \overline{\mathcal{G}} \mid r \le d(z, \Omega) \le R\}.$$

By Theorem 3.6, given $\epsilon > 0$ there is a function $F \in \mathcal{A}_V$ such that $|F(z) - G(z)| < \epsilon$ for all $z \in \Omega$. As in Theorem 4.3, the equation

$$-D^2 h = D^2 F, \quad h\Big|_{\partial \overline{\mathcal{G}}} = 0,$$

may be solved for $h$ using the operator $\mathcal{L}_K$, and $F + h$ will be harmonic. By Corollary 3.7 we may assume that $F(z) = 0$ for $z \in \overline{\mathcal{G}} \setminus N_R(\Omega)$, in which case $f = F + h$ is square integrable on $\overline{\mathcal{G}}$ if $N_R(\Omega)$ has finite volume. □

## 4.2 Self adjoint domains for $-D^2$

As noted earlier, the fact that $S_K$ is bounded below by 0 insures that the Friedrich's extension exists, with a domain which is a subset of $H^1(\mathcal{G})$. In this section we consider self adjoint extensions of the operator $S_K$ which are defined by boundary conditions analogous to the classical Dirichlet and Neumann boundary conditions for $-D^2$ on an interval.

Recall that $\mathcal{A}_\Omega$ is the subalgebra of $\mathcal{A}$ which vanishes on the closed set $\Omega$. Let

$$\mathcal{A}_\Omega^2 = \mathcal{A}_\Omega \cap L^2(\mathcal{G})$$

be the vector space of square integrable functions in $\mathcal{A}_\Omega$, and define a symmetric operator $S_\Omega$ which acts by $-D^2$ on the domain

$$\mathcal{D}_\Omega = \mathcal{D}_K + \mathcal{A}_\Omega^2$$

the linear span of the functions in $\mathcal{D}_K$ together with the square integrable functions in $\mathcal{A}_\Omega$. Note that $\mathcal{D}_\Omega \subset H^1$. Each of these symmetric operators has the quadratic form

$$\langle S_\Omega f, f \rangle = \int (f')^2,$$

and a corresponding Friedrich's extension $\mathcal{L}_\Omega$ whose domain is a subset of $H^1(\mathcal{G})$.

**Theorem 4.5.** *Suppose that $\overline{\mathcal{G}}$ is connected, weakly connected, and has finite diameter. Assume that for $j = 1, 2$, $\Omega(j) \subset \partial\overline{\mathcal{G}}$ is compact, and moreover that for some $R > 0$ the sets*

$$N_R(\Omega(j)) = \{z \in \overline{\mathcal{G}} \mid d(z, \Omega(j)) \le R\}.$$

*are compact with finite volume. If $\Omega(1) \ne \Omega(2)$, then $\mathcal{L}_{\Omega(1)} \ne \mathcal{L}_{\Omega(2)}$.*

*Proof.* The proof of Lemma 3.10 shows that every function $f_1$ in the domain of $\mathcal{L}_{\Omega(1)}$ is continuous on $\overline{\mathcal{G}}$ with $f_1(x) = 0$ for $x \in \Omega(1)$.

Since $\Omega(1)$ and $\Omega(2)$ are distinct, there is an $x \in \Omega(1)$ which is in the complement $\Omega(2)^c$ of $\Omega(2)$. Since $\Omega(2)^c$ is open, there is an $r > 0$ such that

$$\overline{B_r(x)} = \{y \mid d(x, y) \le r\} \subset [\Omega(2)^c \cup N_R(\Omega(1))].$$

Pick $0 < r_1 < r$, and let

$$V = \{z \in \overline{\mathcal{G}} \mid r_1 \le d(z, x) \le r\}.$$

Since $x$ and $V$ are disjoint compact sets, Proposition 3.5 shows that there is a function $f \in \mathcal{A}$ which vanishes on $V$, but $f(x) = 1$. By Theorem 3.7 we may choose $f$ to vanish in the complement of $V$, so $f(y) = 0$ for $y \in \Omega(2)$. Since $N_R(\Omega(1))$ has finite volume, $f \in \mathcal{A}^2_{\Omega(2)}$. Thus $\mathcal{D}_{\Omega(1)} \ne \mathcal{D}_{\Omega(2)}$.

□

If $\mathcal{G}$ is connected and has finite volume, then $\overline{\mathcal{G}}$ is compact, and by Theorem 2.10 is also weakly connected. In this context we have the following corollary of Theorem 4.5.

**Corollary 4.6.** *Suppose $\mathcal{G}$ is connected and has finite volume. If $\Omega(1)$ and $\Omega(2)$ are distinct closed subsets of $\partial\overline{\mathcal{G}}$, then $\mathcal{L}_{\Omega(1)} \ne \mathcal{L}_{\Omega(2)}$.*

In contrast with the previous theorem, larger graphs may have a unique self adjoint extension of $S_K$ with the quadratic form

$$\langle S_\Omega f, f \rangle = \int (f')^2.$$

Recall Proposition 3.13, which guarantees the existence of at least one invertible self adjoint extension of $S_K$ whose domain is a subset of $H^1$.

**Theorem 4.7.** *Suppose $\mathcal{G}$ is compact, connected, and every nonempty open neighborhood of $x$ has infinite volume for every $x \in \partial\overline{\mathcal{G}}$. If $S_K$ has a strictly positive lower bound, then $\mathcal{L}_K$ is the unique self adjoint extension of $S_K$ whose domain is a subset of $H^1$.*

*Proof.* Suppose $\mathcal{L}_1$ is a self adjoint extension of $S_K$ whose domain is a subset of $H^1$. If $\mathcal{L}_1 \neq \mathcal{L}_K$, then there is some $w$ in the domain of $\mathcal{L}_1$, but not in the domain of $\mathcal{L}_K$. Let $g = \mathcal{L}_1 w$.

Since $S_K$ has a strictly positive lower bound, $\mathcal{L}_K$ is invertible. If $v = \mathcal{L}_K^{-1} g$, then the function $u = v - w \neq 0$ is harmonic and in $H^1$. By Theorem 3.2, $u$ extends continuously to $\overline{\mathcal{G}}$.

For each $x \in \partial\overline{\mathcal{G}}$, the condition that every nonempty open neighborhood of $x$ has infinite volume means that the continuous function $u$ must vanish at $x$, since $u \in L^2$. Since $u$ is harmonic, and $\overline{\mathcal{G}}$ is compact, $u$ must be the 0 function. This contradicts the assumption that $\mathcal{L}_1 \neq \mathcal{L}_K$. $\square$

The operators $S_\Omega$ are defined by local boundary conditions in the sense that if $f \in \mathcal{D}_\Omega$ and $\phi \in \mathcal{A}$, then $\phi f \in \mathcal{D}_\Omega$. Here we are using the facts that a function $\phi \in \mathcal{A}$ is constant in a neighborhood of any interior vertex, and all derivatives of $\phi$ are bounded. The invariance of $\mathcal{D}_\Omega$ under multiplication by $\phi \in \mathcal{A}$ also extends to the domains of the operators $\mathcal{L}_\Omega$.

**Theorem 4.8.** *Suppose $f$ is in the domain of $\mathcal{L}_\Omega$ for some closed set $\Omega \subset \partial\overline{\mathcal{G}}$. Then for any $\phi \in \mathcal{A}$, the function $\phi f$ is in the domain of $\mathcal{L}_\Omega$.*

*Proof.* One checks easily that multiplication by $\phi$ is continuous on $L^2(\mathcal{G})$ and on $H^1(\mathcal{G})$. Suppose $f$ is in the domain of $\mathcal{L}_\Omega$, the Friedrich's extension of $S_\Omega$. Since the quadratic form for $S_\Omega$ is

$$Q(f,f) = \langle S_\Omega f, f \rangle = \int_{\mathcal{G}} |f'|^2,$$

$f$ is the limit in $H^1$ of a sequence of functions $f_m \in \mathcal{D}_\Omega$. The functions $\phi f_m$ are also in $\mathcal{D}_\Omega$, and by continuity the sequence $\phi f_m$ converges in $H^1$ to $\phi f$.

We'd like to also establish that $\phi f$ is in the domain of $S_\Omega^*$. For $g \in \mathcal{D}_\Omega$ let

$$I = \langle S_\Omega g, \phi f \rangle = -\int_{\mathcal{G}} g'' \phi f = -\int_{\mathcal{G}} \phi g'' f.$$

The product rule gives

$$I = -\int_{\mathcal{G}} [(\phi g)'' - \phi'' g - 2\phi' g'] f = I_1 + I_2 + I_3.$$

Now $f$ is in the domain of $S_\Omega^*$, and $S_\Omega^* f = -f''$. Since $\phi g \in \mathcal{D}_\Omega$

$$I_1 = -\int_{\mathcal{G}} (\phi g)'' f = -\int_{\mathcal{G}} g\phi f'',$$

while

$$I_2 = \int_{\mathcal{G}} g\phi'' f.$$

For $I_3$, integrate by parts on each edge $e = [a, b]$, obtaining

$$\int_e 2\phi' g' f = 2g\phi' f\Big|_a^b - \int_e 2g(\phi' f)'.$$

Since $\phi \in \mathcal{A}$, $\phi'$ is zero at each vertex, giving

$$I_3 = -\int_{\mathcal{G}} 2g(\phi' f)'.$$

Putting the pieces together and using the product rule gives

$$\langle S_\Omega g, \phi f \rangle = -\int_{\mathcal{G}} g(\phi f)'',$$

so $\phi f$ is in the domain of $S_\Omega^*$. Since $\phi f$ is also in the $H^1$ closure of $\mathcal{D}_\Omega$, the function $\phi f$ is in the domain of $\mathcal{L}_\Omega$.

□

We conclude with a brief discussion of another self adjoint extension of $S_K$ associated with harmonic functions. The link between harmonic functions and self adjoint extensions follows from the fact that the null space of $S_K^*$ is the collection of square integrable harmonic functions on $\mathcal{G}$. Variants of the following result about symmetric operators on a Hilbert space may be found in [6, p. 1396] , [5, p. 210], [9], and the idea is implicit in [1, vol. 2 p. 109].

*Suppose $S$ is symmetric and $N$ is the null space of $S^*$. Then the operator $S_N$ with domain $\mathcal{D}(S) + N$,*

$$S_N(g + h) = Sg, \quad g \in \mathcal{D}(S), \quad h \in N,$$

*is symmetric. If in addition $S$ has closed range, then $S_N$ is self adjoint.*

For graphs $\mathcal{G}$ with finite volume, Theorem 4.3 may often be used to show that $S_K$ has nontrivial or even infinite dimensional deficiency subspaces.

When the symmetric operator is $S_K$ on the unit interval, the operator $S_N$ may be associated with the boundary conditions

$$f'(0) - f'(1) = 0, \quad f(1) - f(0) - f'(0) = 0.$$

Since the domain includes all solutions of $f'' = 0$, it cannot be specified by a pair of independent separated boundary conditions. For $f$ in the domain, rather than the quadratic form $\int_0^1 (f')^2$, integration by parts gives

$$\int_0^1 -f''f = -f'f\Big|_0^1 + \int_0^1 (f')^2 = -(f'(0))^2 + \int_0^1 (f')^2.$$

Suppose $\mathcal{G}$ is a finite graph with at least 2 boundary vertices. Let $v$ be a boundary vertex with the incident edge $e = [a, b]$. The constant function 1 is harmonic, and Theorem 4.3 implies that there is a harmonic function $u$ with $u(v) = 1$, and $u(w) = 0$ at all other boundary vertices $w$. By the maximum principle, $u$ is not constant on $e$. We claim that the harmonic extension $S_N$ cannot have the locality property of Theorem 4.8. Take a function $\phi \in \mathcal{A}$ which is 1 in a small neighborhood of $v$, but which vanishes in a neighborhood of the other vertex of $e$, as well as on every other edge. If Theorem 4.8 were satisfied, then $\phi$ and $\phi u$ would be in the domain of $S_N$. However

$$\int_{\mathcal{G}} \phi''(\phi u) - \phi(\phi u)'' = [\phi'\phi u - \phi(\phi u)']\Big|_a^b \neq 0,$$

and the domain is now too large for $-D^2$ to be symmetric.

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