Spectral Theory for Nonconservative Transmission Line Networks

Robert Carlson
Department of Mathematics
University of Colorado at Colorado Springs
carlson@math.uccs.edu

December 2, 2009

Abstract

The global theory of transmission line networks with nonconservative junction conditions is developed from a spectral theoretic viewpoint. The rather general junction conditions lead to spectral problems for nonnormal operators. The theory of analytic functions which are almost periodic in a strip is used to establish the existence of an infinite sequence of eigenvalues and the completeness of generalized eigenfunctions. Simple eigenvalues are generic. The asymptotic behavior of an eigenvalue counting function is determined. Specialized results are developed for rational graphs.

1 Introduction

The transmission line equations

$$\frac{\partial p}{\partial t} + \frac{1}{\mathbf{C}} \frac{\partial q}{\partial x} = 0. \tag{1.1}$$

$$\frac{\partial q}{\partial t} + \mathbf{A} \frac{\partial p}{\partial x} = 0.$$

provide a basic model for wave propagation in one space dimension. This and related nonlinear systems appear in one dimensional models of arterial blood flow [4, 14, 15, 16], where $q(t,x)$ is the flow in the direction of increasing x, and $p(t,x)$ is the pressure. With $\mathbf{A}$ representing the vessel cross-sectional area divided by the fluid density, and $\mathbf{C}$ being the vessel compliance, both assumed constant between arterial junctions, the equations are derived using conservation of mass and momentum. This system, or a dissipative variant, is frequently used in electrical engineering [12, 13, 17] for describing the propagation of voltage and current along a transmission line.

In many applications of the transmission line equations the natural spatial domain is a network, with multiple finite length transmission lines meeting at junctions. Transmission and reflection at junctions are typically described by boundary conditions or scattering matrices. The description is often local in both space and time, since trying to track even simple initial data through multiple junctions quickly becomes intimidating. Boundary conditions are usually chosen so the evolution is energy conserving. This typical approach to transmission line networks fails to address two important issues: the impact of the global network structure, and the consequences of nonconservative junction conditions. These issues seem particularly important when trying to come to grips with a complex network like the human circulatory system, with its extraordinary number of vessels, junctions, and scales.

This work uses Hilbert space operator and 'quantum graph' methods to treat (1.1) and more general problems on an arbitrary finite network. Losses (and gains) at network junctions are allowed. Although the evolution is generated by a formally skew-adjoint operator, the rather general junction conditions allow for strong domain perturbations away from normal operators. It is thus a pleasant surprise to find that the spectral theory for the generator of the evolution preserves much of the structure found in the skew-adjoint case.

This paper has three main sections. The second section considers nonconservative junction conditions at a single vertex. The focus is on a particular family of junction conditions that, in the fluid flow model, preserve mass but allow pressure losses as a function of the flows at the junction. The third section treats the spectral theory of more general problems, realized on directed finite networks with invertible junction scattering matrices. The eigenvalues associated to these systems are the roots of an entire function of the complex eigenvalue parameter which are almost periodic in a vertical strip. Aspects of the theory of analytic almost periodic functions play an important role in the analysis. Among other results, the eigenvalues are shown to be generically simple, to satisfy a 'Weyl' estimate relating asymptotics of the eigenvalue counting function to the graph 'volume', and to have a complete set of generalized eigenfunctions. The fourth section treats the case when network edge lengths are rational multiples of a common value. Here the link between the differential operator eigenvalues and the eigenvalues of an aggregate junction scattering matrix are more explicit. In addition, the completeness result for generalized eigenfunctions can be sharpened to give a Reisz basis of generalized eigenfunctions. Returning to the more specialized models of the second section, the existence of eigenvalues with real part 0 is shown to depend on rational relations among cycle lengths of the network.

The methods employed here are related to ideas in the 'quantum graph' literature. We will often employ graph terminology, using edges instead of transmission lines, and vertices in place of junctions. Some of the problems and techniques of this work have published antecedents. The problem of finding vertex conditions leading to an energy conserving or energy dissipating evolutions was previously considered in [2, 3]. We will make use of a scheme for replacing an undirected graph with a directed cover; this is at least implicit in [9]. Characterizations of a different class of contraction semigroups on graphs is considered in [8].

For notational convenience, $[x_1, \dots, x_N]$ will represent the column vector $(x_1, \dots, x_N)^T$.

2 Nonconservative network junctions

If N transmission lines $e_1, \dots, e_N$ meet at a junction (Figure A), the equations (1.1) describing propagation along a line are supplemented by junction conditions. This section considers an extension of the standard junction conditions to allow for energy loss (or gain) at the junctions. Anticipating more general developments, these nonconservative junction conditions are recast as junction scattering matrices for the propagation of incoming signals into outgoing signals.

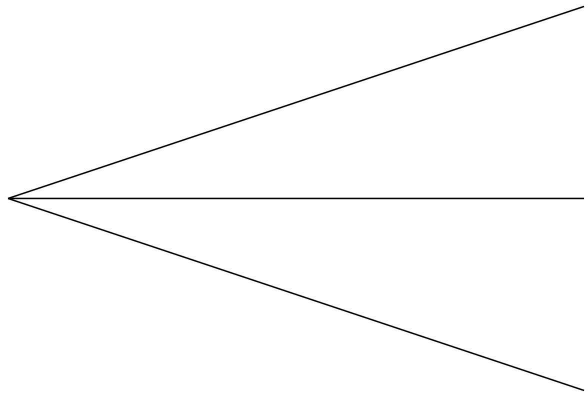

Figure A: A transmission line junction

For each transmission line (or edge) e_n, the parameters $\mathbf{A}_n$ and $\mathbf{C_n}$ from (1.1) are assumed positive, and constant on e_n, but possibly varying with n. Assume that e_n has a local coordinate x which increases with distance from the junction. Introduce the wave speed

$$c_n = \sqrt{\mathbf{A}_n/\mathbf{C_n}},$$

the impedance [4, p. 157]

$$Z_n = c_n/\mathbf{A}_n, \tag{2.1}$$

and the functions

$$R_n = \frac{1}{\sqrt{Z_n}}p_n + \sqrt{Z_n}q_n, \quad L_n = \frac{1}{\sqrt{Z_n}}p_n - \sqrt{Z_n}q_n. \tag{2.2}$$

In these new variables, each equation (1.1) is equivalent to a diagonal system

$$\frac{\partial R_n}{\partial t} = -c_n\frac{\partial R_n}{\partial x}, \quad \frac{\partial L_n}{\partial t} = c_n\frac{\partial L_n}{\partial x}, \quad n = 1, \ldots, N. \tag{2.3}$$

On each edge e_n solutions $R_n(t,x)$, respectively $L_n(t,x)$, are simply traveling waves moving at constant speed in the direction of increasing, respectively decreasing, x.

Propagation through a junction v is typically described by junction conditions or junction scattering matrices. The following family of junction conditions seems especially attractive as a model for energy dissipation.

$$\sum_{n=1}^{N(v)} q_n(v) = 0, \tag{2.4}$$

$$p_n(v) - p_{n-1}(v) = \epsilon_v(q_n(v) - q_{n-1}(v)), \quad \epsilon_v \le 0, \quad n = 2, \dots, N.$$

The first condition is the natural requirement of mass conservation (respectively charge conservation) for transmission line models of fluid flow (respectively voltage and current flow). Conventional models [15, 16] typically use $\epsilon = 0$, so that pressures (or voltages) agree at a junction. Notice that (2.4) implies the further conditions

$$p_m(v) - p_n(v) = \epsilon_v(q_m(v) - q_n(v)).$$

With a suitable inner product, the spatial part of (1.1) is formally skew adjoint. For vector functions $[P, Q]$ and $[U, V]$ with $2N$ components $P = [p_1, \dots, p_N]$, $Q = [q_1, \dots, q_N]$, etc., define the inner product

$$\langle [P, Q], [U, V] \rangle = \sum_{n=1}^{N} \int [\mathbf{C}_n p_n \overline{u_n} + \frac{1}{\mathbf{A}_n} q_n \overline{v_n}].$$

If L acts by

$$L[P, Q] = [\frac{-1}{\mathbf{C}_1} \partial_x q_1, \dots \frac{-1}{\mathbf{C}_N} \partial_x q_N, -\mathbf{A}_1 \partial_x p_1, \dots, -\mathbf{A}_N \partial_x p_N],$$

and the functions vanish at the upper integration limit (as they might with a partition of unity), then integration by parts gives

$$\langle L[P, Q], [U, V] \rangle + \langle [P, Q], L[U, V] \rangle = \sum_{n=1}^{N} [q_n \overline{u_n}(v) + p_n \overline{v_n}(v)].$$

This last boundary term vanishes if $[P, Q]$ and $[U, V]$ satisfy (2.4) with $\epsilon_v = 0$.

If $\epsilon_v < 0$ and the conditions (2.4) hold, then L is formally dissipative, since the associated quadratic form has

$$\langle L[P, Q], [P, Q] \rangle + \langle [P, Q], L[P, Q] \rangle = 2\Re(\langle L[P, Q], [P, Q] \rangle) \tag{2.5}$$

$$= \sum_{n=1}^{N} [q_n \overline{p_n}(v) + p_n \overline{q_n}(v)] = \sum_{n=1}^{N} [q_n \overline{(p_n - p_1)}(v) + (p_n - p_1) \overline{q_n}(v)]$$

$$= \epsilon_v \sum_{n=1}^{N} [q_n \overline{(q_n - q_1)}(v) + (q_n - q_1) \overline{q_n}(v)] = 2\epsilon_v \sum_{n=1}^{N} |q_n(v)|^2.$$

2.1 Junction scattering matrices

By shifting to the variables R_n and L_n introduced in (2.2) the boundary conditions (2.4) can be recast as junction scattering matrices. Recall that an outgoing solution $R_n(t,x)$ for (2.3) is a traveling wave moving away from the junction v, while an incoming solution $L_n(t,x)$ is moving toward v.

In terms of ingoing and outcoming data, and suppressing the time dependence, the conditions (2.4) are

$$\sum_{n=1}^{N} \frac{1}{\sqrt{Z_n}}(R_n(v) - L_n(v)) = 0, \tag{2.6}$$

$$\sqrt{Z_n}(R_n(v) + L_n(v)) - \sqrt{Z_{n-1}}(R_{n-1}(v) + L_{n-1}(v))$$
$$= \frac{\epsilon_v}{\sqrt{Z_n}}(R_n(v) - L_n(v)) - \frac{\epsilon_v}{\sqrt{Z_{n-1}}}(R_{n-1}(v) - L_{n-1}(v)), \quad n = 2, \dots, N.$$

For notational convenience define

$$z_n^{\pm} = \sqrt{Z_n} \pm \frac{\epsilon}{\sqrt{Z_n}}, \quad n = 1, \dots, N, \tag{2.7}$$

the $N \times N$ matrices

$$\mathcal{Z}^{\pm} = \begin{pmatrix} 1/\sqrt{Z_1} & 1/\sqrt{Z_2} & 1/\sqrt{Z_3} & \dots & 1/\sqrt{Z_N} \\ -z_1^{\pm} & z_2^{\pm} & 0 & \dots & 0 \\ 0 & -z_2^{\pm} & z_3^{\pm} & \dots & 0 \\ 0 & \vdots & \vdots & \dots & 0 \\ 0 & 0 & \dots & -z_{N-1}^{\pm} & z_N^{\pm} \end{pmatrix},$$

and

$$\mathcal{J} = \mathrm{diag}[1, -1, \dots, -1].$$

Introduce the vector functions

$$\mathcal{L} = [L_1, \dots, L_N], \quad \mathcal{R} = [R_1, \dots, R_N]. \tag{2.8}$$

Separating the incoming and outgoing data, the equations (2.6) may then be written

$$\mathcal{Z}^{-}\mathcal{R}(v) = \mathcal{J}\mathcal{Z}^{+}\mathcal{L}(v). \tag{2.9}$$

Proposition 2.1. *If $|\epsilon| < Z_n$ for $n = 1, \dots, N$, then the matrices $\mathcal{Z}^{\pm}$ are invertible.*

Proof. The argument is the same for the two matrices $\mathcal{Z}^{\pm}$, so consider $\mathcal{Z}^{+}$. The condition $|\epsilon| < Z_n$ for $n = 1, \dots, N$ implies $z_n^+ > 0$. Alter the first row of $\mathcal{Z}^+$ to obtain a new matrix

$$\mathcal{Z}_1^+ = \begin{pmatrix} 1/z_1^+ & 1/z_2^+ & 1/z_3^+ & \dots & 1/z_N^+ \\ -z_1^+ & z_2^+ & 0 & \dots & 0 \\ 0 & -z_2^+ & z_3^+ & \dots & 0 \\ 0 & \vdots & \vdots & \dots & 0 \\ 0 & 0 & \dots & -z_{N-1}^+ & z_N^+ \end{pmatrix},$$

Observe that all rows are nonzero vectors, the first row is orthogonal to the other rows, and rows $2, \dots, N$ are clearly independent. Since its rows are independent, $\mathcal{Z}_1^+$ is invertible.

Shifting back to the matrix $\mathcal{Z}^+$, notice that its first row is nonzero, and not orthogonal to the first row of $\mathcal{Z}_1^+$, since the dot product of the first rows is positive. This makes all rows of $\mathcal{Z}^+$ independent, as desired.

□

Proposition 2.2. *If the matrices $\mathcal{Z}^{\pm}$ are invertible, the junction scattering matrix*

$$T_\epsilon = (\mathcal{Z}^-)^{-1} J \mathcal{Z}^+ \tag{2.10}$$

has an eigenvector $V = [1/\sqrt{Z_1}, \dots, 1/\sqrt{Z_N}]$, with eigenvalue $\mu = 1$.

The subspace $V^\perp$ is an invariant subspace for T_ϵ.

Proof. To see that V satisfies

$$\mathcal{Z}^- V = J \mathcal{Z}^+ V$$

it suffices to take $R_n = L_n = 1/\sqrt{Z_n}$ and check that (2.6) is satisfied.

For all $W \in V^\perp$ the vectors $\mathcal{Z}^- W$ and $J\mathcal{Z}^+ W$ have first component 0. Since $\mathcal{Z}^-$ is invertible, $(\mathcal{Z}^-)^{-1}$ carries the vectors with first component 0 onto $V^\perp$. Thus

$$T_\epsilon W = (\mathcal{Z}^-)^{-1} J \mathcal{Z}^+ W \in V^\perp, \quad W \in V^\perp.$$

□

The next proposition confirms that the junction scattering matrix is unitary [2] in the standard case $\epsilon = 0$.

Proposition 2.3. *The matrix T_0 is orthogonal and real symmetric, with* 1 *being an eigenvalue of multiplicity* 1*, and* -1 *an eigenvalue of multiplicity* $N-1$*.*

Proof. In case $\epsilon = 0$, let $\mathcal{Z} = \mathcal{Z}^+ = \mathcal{Z}^-$. For $n = 1, \dots, N$ let V_n denote the transpose of the $n-th$ row of $\mathcal{Z}$. Then

$$\mathcal{J}\mathcal{Z}V_1 = \mathcal{Z}V_1, \quad \mathcal{J}\mathcal{Z}V_n = -\mathcal{Z}V_n, \quad n = 2, \dots, N,$$

and V_1 is orthogonal to the independent vectors $V_2, \dots, V_N$. Since V_1 is an eigenvector of $\mathcal{Z}^{-1}\mathcal{J}\mathcal{Z}$ with eigenvalue 1, and $V_2, \dots, V_N$ are eigenvectors with eigenvalue -1, the matrix $\mathcal{Z}^{-1}\mathcal{J}\mathcal{Z}$ is orthogonal and real symmetric. □

The vertex form (2.5) gives the following information.

Proposition 2.4. *Suppose T_ϵ exists for some $\epsilon < 0$. Then all eigenvalues of $T_\epsilon^* T_\epsilon$ lie in the interval* $[0, 1]$.

Proof. Using (2.2) together with (2.5), the boundary condition (2.4) gives

$$2\epsilon_v \sum_{n=1}^N |q_n(v)|^2 = \sum_{n=1}^N [q_n \overline{p_n}(v) + p_n \overline{q_n}(v)] \tag{2.11}$$

$$= \sum_{n=1}^N [(R_n - L_n)\overline{(R_n + L_n)} + (R_n + L_n)\overline{(R_n - L_n)}]$$

$$= 2\sum_{n=1}^N [|R_n|^2 - |L_n|^2] = 2\|T_\epsilon \mathcal{L}\|^2 - 2\|\mathcal{L}\|^2 = 2(T_\epsilon^* T_\epsilon - I)\mathcal{L} \cdot \mathcal{L}.$$

By assumption, $\epsilon < 0$, so the eigenvalues ν of $T_\epsilon^* T_\epsilon$, which are always non-negative, also satisfy $\nu \le 1$. □

Proposition 2.5. *If $-Z_n < \epsilon_v < 0$ for $n = 1, \dots, N$, then the junction scattering map $T_\epsilon(v)$ has only real eigenvalues. Counted with algebraic multiplicity, $N-1$ eigenvalues μ_n lie in the interval*

$$-1 < \mu_n < \max_n(-1 - 2\epsilon_v/(Z_n - \epsilon_v)).$$

Proof. The bounds on ϵ_v imply that T_ϵ is defined and invertible by (2.7) and Proposition 2.1. Suppose $\mathcal{L} = (L_1, \ldots, L_N)$ is an eigenvector with eigenvalue $\mu \neq 1$. Then $\mathcal{R} = \mu\mathcal{L}$ and equations (2.6) become

$$\sum_{n=1}^{N} \frac{\mu - 1}{\sqrt{Z_n}} L_n = 0, \tag{2.12}$$

$$[\sqrt{Z_n}(1+\mu) + \frac{\epsilon_v(1-\mu)}{\sqrt{Z_n}}]L_n = [\sqrt{Z_{n-1}}(1+\mu) + \frac{\epsilon_v(1-\mu)}{\sqrt{Z_{n-1}}}]L_{n-1}.$$

With

$$f_n(\mu) = \sqrt{Z_n}(1+\mu) + \frac{\epsilon_v(1-\mu)}{\sqrt{Z_n}} = \frac{1}{\sqrt{Z_n}}[Z_n(1+\mu) + \epsilon_v(1-\mu)],$$

the above recursion for L_n can be written as

$$f_n(\mu)L_n = f_{n-1}(\mu)L_{n-1},$$

which has a solution

$$L_n = \prod_{k \neq n} f_k(\mu). \tag{2.13}$$

Plugging this into the first condition of (2.12) gives

$$\sum_{n=1}^{N} \frac{\mu - 1}{\sqrt{Z_n}} \prod_{k \neq n} f_k(\mu) = 0.$$

That is, a desired eigenvector $\mathcal{L} = (L_1, \ldots, L_N)$ is given by (2.13) if $\mu \neq 1$ is a root of

$$g(\mu) = \sum_{n=1}^{N} \prod_{k \neq n} [Z_k(1+\mu) + \epsilon_v(\mu - 1)] = 0,$$

and $\mathcal{L} \neq 0$.

Notice first that $g(-1) \neq 0$ for $\epsilon_v \neq 0$. If

$$\sigma_n = \frac{-\epsilon_v - Z_n}{Z_n - \epsilon_v} = -1 - \frac{2\epsilon_v}{Z_n - \epsilon_v},$$

then $-1 < \sigma_n < 0$, $f_n(\sigma_n) = 0$, and

$$g(\sigma_n) = \prod_{k \neq n} [Z_k(1+\sigma_n) + \epsilon_v(\sigma_n - 1)].$$

Among the N-tuples $(Z_1, \ldots, Z_N)$ with positive components there is a dense open set giving distinct values σ_n, which may be assumed to be ordered so that $-1 > \sigma_1 > \cdots > \sigma_N$. The numbers $g(\sigma_n)$ then have alternating signs, and g has $N-1$ distinct real roots $-1 < \mu_n < 0$, with $\mathcal{L} \neq 0$. The statement of the theorem then follows for all positive values of $\{Z_n\}$ by continuity. $\square$

2.2 A directed graph framework

The next step in this analysis of nonconservative transmission line networks is to place such a network in a somewhat more general context. Suppose $\mathcal{G}^u$ is an undirected finite graph. We would like to consider equations of the form (1.1) or equivalently (2.3) on each edge of $\mathcal{G}^u$, subject to the junction conditions (2.4), or alternatively (2.6). A difficulty is that q, or the functions R and L, depend on the edge direction. For many networks there is no preferred choice of edge directions, and arbitrary choices can lead to considerable notational confusion.

A productive alternative is to replace the undirected graph $\mathcal{G}^u$ with a directed graph $\mathcal{G}$ having twice as many edges. Given two adjacent vertices v, w in $\mathcal{G}^u$, replace the undirected edge e_0 joining them with two directed edges, e_1 from v to w, and e_2 from w to v. If e_0 has length l, so do e_1 and e_2. If x is a local coordinate for e_0, increasing from v to w, choose coordinates $X = x$ for e_1 and $X = l - x$ for e_2, so X increases in the edge direction for all directed edges. The vector of functions (R, L) on an undirected edge e_0 can now be replaced by scalar functions $f_1(t, X) = R(t, x)$ and $f_2(t, X) = L(t, l - x)$ on the directed edges e_1, e_2. With respect to such local coordinates the system (2.3) becomes

$$\frac{\partial f_e}{\partial t} = -c_e \frac{\partial f_e}{\partial X}, \tag{2.14}$$

on the edges e of the new directed graph $\mathcal{G}$. Solutions on all directed edges are traveling waves moving at speed c_e in the direction of increasing x.

To complete the global description, introduce the junction scattering matrices $T(v)$, having the form (2.10). The parameter ϵ may vary from vertex to vertex, but we assume that $T(v)$ is invertible.

Suppose that λ is an eigenvalue with eigenfunction F for the spatial part (or right hand side of) system (2.14), subject to a set of junction scattering conditions, $F_{out}(v) = T(v) F_{in}(v)$. Then (2.14) will have solutions

$$\mathcal{F}(t, x) = e^{\lambda t} F(x).$$

For the undirected edges $e_1, \ldots, e_N$ incident on a vertex v in $\mathcal{G}^u$, using local edge coordinates which are the distance from v, the solution $\mathcal{F}(t,x)$ may be written as a vector solution of (2.3)

$$e^{\lambda t}[\mathcal{R}, \mathcal{L}].$$

Using the substitution (2.2), we find, in these local coordinates, separated solutions of (1.1)

$$p_n(t,x) = e^{\lambda t}\sqrt{Z_n}[R_n(x) + L_n(x)], \quad q_n(t,x) = e^{\lambda t}\frac{1}{\sqrt{Z_n}}[R_n(x) - L_n(x)].$$

Before proceeding with a general discussion of eigenvalue problems for derivative operators on directed graphs, a simple example is presented as a preview of the issues that will arise. In case $\epsilon(v) = 0$ for all junction conditions (2.4) or (2.10), the spatial operator appearing in (2.14) can be interpreted as a skew-adjoint Hilbert space operator with compact resolvent. Powerful abstract results are then at our disposal. Handling the case $\epsilon \ne 0$ requires new methods.

For the example, the network consists of two edges of length 1 joined at a junction. Rather than treating a system of equations on each edge, replace each of the original edges E_1, E_2 with two directed edges, resulting in 4 directed edges $e_1, \ldots, e_4$, with e_1 and e_2 directed from the junction outward, and e_3 and e_4 directed from the junction inward. The original pair of waves $(R(t,x), L(t,x))$ on undirected edges can now be thought of as a single scalar function $F(t,x)$ on directed edges.

Choose local coordinates identifying directed edges with $[0,1]$, so the local coordinate increases in the direction of the edge. A function on the directed graph may be identified with a vector function $f : [0,1] \to \mathbb{C}^4$ and the spatial operator has the vector form

$$Df = \mathrm{diag}[c_1, c_2, c_1, c_2]f'.$$

At the boundary vertices where the edges E_n were not joined, the conditions (2.6) become $R_n(v) = L_n(v)$. In terms of vector functions

$$F_1 = [f_1, f_2], \quad F_2 = [f_3, f_4],$$

the junction and boundary conditions (2.6) may be combined as

$$\begin{pmatrix} F_1(0) \\ F_2(0) \end{pmatrix} = T \begin{pmatrix} F_1(1) \\ F_2(1) \end{pmatrix}.$$

Assuming that $Z_j = c_j$, the amalgam of junction scattering matrices is

$$T = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{c_2 - c_1 - 2\epsilon}{c_1 + c_2 - 2\epsilon} & \frac{2\sqrt{c_1 c_2}}{c_1 + c_2 - 2\epsilon} & 0 & 0 \\ \frac{2\sqrt{c_1 c_2}}{c_1 + c_2 - 2\epsilon} & \frac{c_1 - c_2 - 2\epsilon}{c_1 + c_2 - 2\epsilon} & 0 & 0 \end{pmatrix}.$$

An eigenfunction with eigenvalue λ will have $c_n f_n' = \lambda f_n$, and $c_n f_{N+n}' = \lambda f_{N+n}$, for $n = 1, 2$, so if $w_n = 1/c_n$ then

$$f_n = \alpha_n \exp(\lambda w_n x), \quad f_{n+2} = \alpha_{n+2} \exp(\lambda w_n x).$$

The boundary conditions give

$$[\alpha_1, \dots, \alpha_4] = T \exp(\lambda W)[\alpha_1, \dots, \alpha_4].$$

In this example

$$\exp(\lambda W) = \mathrm{diag}[e^{w_1 \lambda}, e^{w_2 \lambda}, e^{w_1 \lambda}, e^{w_2 \lambda}].$$

Eigenvalues are solutions of the equation

$$\det[I - T \exp(\lambda W)] = 0,$$

or

$$\det \begin{pmatrix} -1 & 0 & e^{\lambda w_1} & 0 \\ 0 & -1 & 0 & e^{\lambda w_2} \\ \frac{c_2 - c_1 - 2\epsilon}{c_1 + c_2 - 2\epsilon} e^{\lambda w_1} & \frac{2\sqrt{c_1 c_2}}{c_1 + c_2 - 2\epsilon} e^{\lambda w_2} & -1 & 0 \\ \frac{2\sqrt{c_1 c_2}}{c_1 + c_2 - 2\epsilon} e^{\lambda w_1} & \frac{c_1 - c_2 - 2\epsilon}{c_1 + c_2 - 2\epsilon} e^{\lambda w_2} & 0 & -1 \end{pmatrix}$$

$$= 1 - \beta_1 e^{2\lambda w_1} - \beta_4 e^{2\lambda w_2} + e^{2\lambda(w_1 + w_2)}[\beta_1 \beta_4 - \beta_2 \beta_3] = 0,$$

where

$$\beta_1 = \frac{c_2 - c_1 - 2\epsilon}{c_1 + c_2 - 2\epsilon}, \quad \beta_4 = \frac{c_1 - c_2 - 2\epsilon}{c_1 + c_2 - 2\epsilon}, \quad \beta_2 = \beta_3 = \frac{2\sqrt{c_1 c_2}}{c_1 + c_2 - 2\epsilon}$$

The eigenvalues λ are thus the roots of an entire function which is almost periodic along vertical lines. It is straightforward to see that such roots must lie in a vertical strip in the complex plane, although some complex analysis seems to be required to show there must be infinitely many. Also notice that if there are integers m_1, m_2, with $w_1 = m_1 r$ and $w_2 = m_2 r$, then the equation for eigenvalues can be written as a polynomial equation in $\exp(\lambda r)$, simplifying considerably the structure of the set of solutions. The extension of such issues and other features to general networks will be considered in the subsequent sections.

3 Derivative Operators on Directed Graphs

In the last section we discussed alternative formulations of the transmission line equations, focusing mainly on a family of generalized junction conditions and some of the features of the corresponding junction scattering matrices. We now consider eigenvalues for these problems, as well as more general problems on networks.

Assume that $\mathcal{G}$ is a directed graph with a finite vertex set and a finite edge set. Each vertex appears in at least one edge, and for convenience loops and multiple edges between vertex pairs are not considered directly, although these can be incorporated by adding vertices. A vertex v of $\mathcal{G}$ will have an equal number $\delta(v)$ of incoming and outgoing edges. Each edge has a positive finite length or weight, and for notational convenience the edges are assumed numbered e_n, $n = 1, \dots, N$. Directed edges e_n of length l_n may be identified with a real interval $e_n = [a_n, b_n]$ of the same length. The Euclidean interval length and Lebesgue measure extend in a standard way to $\mathcal{G}$.

Treatment of the system (2.14) is simplified by a linear change of variables identifying each directed edge with $[0, 1]$, so that edges exit a vertex at 0 and enter at 1. The formal operator $c_n \partial/\partial X$ is then replaced by $\frac{1}{w_n}\partial/\partial x$, with $w_n = l_n/c_n$. Introduce the weighted Hilbert space

$$\mathbb{H} = \oplus_n L^2([0,1], w_n), \quad w_n = \frac{l_n}{c_n},$$

with inner product

$$\langle F, G \rangle = \sum_n \int_0^1 f_n(x)\overline{g_n(x)}\; w_n \; dx, \quad F = [f_1, \dots, f_N], \quad G = [g_1, \dots, g_N],$$

where $[f_1, \dots, f_N]$ denotes the column vector $(f_1, \dots, f_N)^T$. Let $\mathcal{D}$ be the operator acting formally by

$$\mathcal{D}[f_1, \dots, f_N] = [\frac{1}{w_1}\frac{\partial f_1}{\partial x}, \dots, \frac{1}{w_N}\frac{\partial f_N}{\partial x}]. \tag{3.1}$$

To study $\mathcal{D}$ as a Hilbert space operator, start with the maximal operator D_{max}, whose domain consists of all $F : [0, 1] \to \mathbb{C}^N$ with absolutely continuous components, and whose derivatives are in $L^2[0, 1]$. Standard results show that $\mathcal{D}_{max}$ is a Fredholm operator of index N.

The domain of $\mathcal{D}$ is determined by invertible scattering matrices. For a continuous function $F : \mathcal{G} \to \mathbb{C}$, let $F_i(v)$, respectively $F_o(v)$, be the $\delta(v)$-tuple of values of F at $x = 1$, respectively $x = 0$, for incoming, respectively outgoing edges at v. Given invertible matrices

$$T(v) : \mathbb{C}^{\delta(v)} \to \mathbb{C}^{\delta(v)} \tag{3.2}$$

at the vertices, the domain of the operator D then consists of those functions in the domain of D_{max} which satisfy the junction conditions

$$F_o(v) = T(v)F_i(v), \quad v \in \mathcal{V}. \tag{3.3}$$

There are N independent boundary conditions, so $\mathcal{D}$ is Fredholm with index 0. (See for instance [7, pp. 145,169,188,272].) If each $T(v)$ is unitary, then $\mathcal{D}$ will be skew-adjoint [2].

3.1 Resolvent

The conditions (3.3) defining the domain of $\mathcal{D}$ may be written as a single linear system

$$f_m(0) - \sum_{n=1}^{N} T_{mn} f_n(1) = 0, \quad m = 1, \dots, N.$$

Using the matrix $T = (T_{mn})$, an explicit formula for the resolvent

$$R_{\mathcal{D}}(\lambda) = [\mathcal{D} - \lambda I]^{-1}$$

is available. Let $W = \mathrm{diag}[w_1, \dots, w_N]$ denote the diagonal matrix of edge weights and let $\exp(x\lambda W) = \mathrm{diag}[\exp(x\lambda w_1), \dots, \exp(x\lambda w_N)]$. Solutions of

$$W^{-1}\partial_x Y - \lambda Y = f, \quad Y = [y_1, \dots, y_N], \quad f = [f_1, \dots, f_N],$$

have the form

$$Y(x, \lambda) = \exp(\lambda x W)K(\lambda) + \int_0^x \exp(\lambda[x - t]W)W f(t)\, dt, \quad K(\lambda) \in C^N.$$

Satisfaction of the boundary conditions $Y(0) = TY(1)$ requires

$$K(\lambda) = [T^{-1} - \exp(\lambda W)]^{-1} \int_0^1 \exp(\lambda[1 - t]W)W f(t)\, dt.$$

After some elementary manipulations,

$$R_{\mathcal{D}}(\lambda)f(x) \tag{3.4}$$
$$= \exp(\lambda x W)\Big([\exp(-\lambda W)T^{-1} - I]^{-1} + I\Big)\int_0^x \exp(-\lambda t W)Wf(t)\ dt$$
$$+ \exp(\lambda x W)[T^{-1} - \exp(\lambda W)]^{-1}\int_x^1 \exp(\lambda[1-t]W)Wf(t)\ dt.$$

Introduce the characteristic function

$$\chi(\lambda) = \det[T^{-1} - \exp(W\lambda)].$$

The resolvent is bounded and compact if $[T^{-1} - \exp(\lambda W)]^{-1}$ exists, or equivalently if $\chi(\lambda) \neq 0$. The next result addresses resolvent bounds.

Theorem 3.1. *The operator $\mathcal{D}$ has compact resolvent $R_{\mathcal{D}}(\lambda)$, with the spectrum contained in a vertical strip $-A \le \Re(\lambda) \le A$. For A sufficiently large the resolvent satisfies*

$$\|R_{\mathcal{D}}(\lambda)\| \le \frac{C}{|\Re(\lambda)|}. \tag{3.5}$$

In addition, the resolvent $R_{\mathcal{D}}(\lambda)$ exists and is uniformly bounded on

$$B_\delta = \{-A \le \Re(\lambda) \le A, |\chi(\lambda)| \ge \delta > 0\}.$$

Proof. Since

$$[T^{-1} - \exp(\lambda W)]^{-1} = [I - T\exp(\lambda W)]^{-1}T,$$

the resolvent set includes the half plane where (using the usual matrix norm)

$$\|\exp(\lambda W)\| = \max_n(|\exp(\lambda w_n)|) < \|T\|^{-1}, \quad \Re(\lambda) < 0.$$

Similarly,

$$[T^{-1} - \exp(\lambda W)]^{-1} = [\exp(-\lambda W)T^{-1} - I]^{-1}\exp(-\lambda W),$$

so the resolvent is bounded if

$$\|\exp(-\lambda W)\| = \max_n(|\exp(-\lambda w_n)|) < \|T^{-1}\|^{-1}, \quad \Re(\lambda) > 0.$$

To establish (3.5), pick A sufficiently large, and consider λ in the half plane $\Re(\lambda) \geq A$. By (3.4), $R_{\mathcal{D}}(\lambda) f(x) = I_1 + I_2$, with

$$I_1 = \exp(\lambda x W)\Big([\exp(-\lambda W)T^{-1} - I]^{-1} + I\Big) \int_0^x \exp(-\lambda t W) W f(t)\ dt,$$

and

$$I_2 = \exp(\lambda x W)[T^{-1} - \exp(\lambda W)]^{-1} \int_x^1 \exp(\lambda[1-t]W) W f(t)\ dt.$$

The geometric series gives

$$[\exp(-\lambda W)T^{-1} - I]^{-1} + I = \exp(-\lambda W) T^{-1} B(\lambda)$$

where $B(\lambda)$ denotes a matrix valued function bounded for $\Re(\lambda) \geq A$. Thus

$$I_1 = \exp(-\lambda(1-x)W) B(\lambda) \int_0^x \exp(-\lambda t W) W f(t)\ dt.$$

For each x the Cauchy-Schwarz inequality gives the componentwise bound

$$|\int_0^x \exp(-\lambda t w_n) w_n f_n(t)\ dt| \leq C \|f_n\| / |Re(\lambda)|^{1/2},$$

so

$$\|I_1\| \leq C \|f_n\| / |Re(\lambda)|.$$

Elementary manipulations give

$$I_2 =$$

$$\exp(\lambda x W)[T^{-1} - \exp(\lambda W)]^{-1} \exp(\lambda[1-x]W) \int_x^1 \exp(\lambda[x-t]W) W f(t)\ dt$$

$$= [\exp(-\lambda(1-x)W) T^{-1} \exp(-\lambda x W) - I]^{-1} \int_x^1 \exp(\lambda[x-t]W) W f(t)\ dt,$$

so the method used to bound I_1 may be used for I_2 to complete estimate (3.5). A similar argument applies in the half plane $\Re(\lambda) \leq -A$.

The usual formula for a matrix inverse

$$[T^{-1} - \exp(W\lambda)]^{-1} = \frac{1}{\chi(\lambda)} adj(T^{-1} - \exp(W\lambda)),$$

where *adj* denotes the transposed cofactor matrix, together with (3.4), shows that the resolvent will be uniformly bounded if $\lambda \in B_\delta$.

$\square$

3.2 Eigenvalues

The system of equations $W^{-1}\partial_x Y = \lambda Y$ has a nontrivial solution satisfying the boundary conditions $Y(0) = TY(1)$ if and only if the characteristic function has a root at λ,

$$\chi(\lambda) = \det[T^{-1} - \exp(W\lambda)] = 0,$$

so these roots are the spectrum of $\mathcal{D}$. Since $W = \mathrm{diag}[w_1, \dots, w_N]$, expansion of the determinant gives

$$\chi(\lambda) = (-1)^N \exp(\lambda r_0) + \sum c_j \exp(\lambda r_j), \tag{3.6}$$

$$0 \le r_J < \cdots < r_1 < r_0 = \sum_{n=1}^{N} w_n.$$

Taking the limit as $\Re(\lambda) \to -\infty$ shows that $r_J = 0$ and $c_J = \det(T^{-1}) \ne 0$.

Fix positive numbers $0 = r_J < \cdots < r_0 = \sum_n w_n$, and introduce the class of functions $\mathcal{E}$ which have the form

$$\sum_{j=0}^{J} c_j \exp(\lambda r_j), \quad c_j \in \mathbb{C}, \quad |c_0| = 1, \quad c_J \ne 0.$$

These are entire functions which are almost periodic ([10, pp. 264–273], and for more detail [6]) in any strip

$$S_A = \{\lambda \in \mathbb{C}, -A \le \Re(\lambda) \le A < \infty\}.$$

For $\chi(\lambda) \in \mathcal{E}$, direct estimation shows that for A large enough, the strip S_A contains all roots of $\chi(\lambda)$ in its interior. Let C_A be the set of bounded continuous complex valued functions on S_A, equipped with the norm

$$\|f(\lambda)\|_A = \sup_{\lambda \in S_A} |f(\lambda)|.$$

Given any $\delta > 0$, and letting λ_j be the roots of $\chi(\lambda)$, define

$$S_\delta = \{\lambda \in S_A, \inf_k |\lambda - \lambda_k| \ge \delta\}$$

to be the set of points with distance at least δ from any root.

The next result summarizes important properties of functions in $\mathcal{E}$, incuding the characteristic function $\chi(\lambda)$. Proofs for the more general setting of almost periodic functions in a strip may be found in [10, pp. 264-9]. For completeness we sketch the arguments.

Lemma 3.2. *Suppose $\chi(\lambda) \in \mathcal{E}$. For $\tau \in \mathbb{R}$, the set of all translates $\chi(\lambda + i\tau)$ has compact closure in C_A. The number of roots of $\chi(\lambda)$, counted with multiplicity, in a rectangle*

$$B = \{\lambda \in \mathbb{C}, -A \le \Re(\lambda) \le A, \tau \le \Im(\lambda) \le \tau + 1\}$$

is bounded by some number N_R which is independent of τ. There is a number $C(\delta) > 0$ such that $|\chi(\lambda)| \ge C(\delta)$ for all $\lambda \in S_\delta$.

Proof. Translates of $\chi(\lambda)$ are

$$\chi(\lambda + i\tau) = c_0 \exp(i\tau r_0) \exp(\lambda r_0) + \sum c_j \exp(i\tau r_j) \exp(\lambda r_j).$$

The $J+1$ coefficients $c_j \exp(i\tau r_j)$ remain in a bounded subset of $\mathbb{C}^{J+1}$, which is the essential point for compactness.

Suppose the bound N_R for the number of roots did not exist. Select a sequence of translates $f_j(\lambda) = \chi(\lambda + i\tau_j)$ with f_j having at least j roots in

$$B_0 = \{\lambda \in \mathbb{C}, -A \le \Re(\lambda) \le A, 0 \le \Im(\lambda) \le 1\}.$$

The sequence $\{f_j\}$ may be assumed to converge uniformly on compact subsets of $\mathbb{C}$ to an entire function $g(\lambda)$ with the same form as $\chi(\lambda)$. In particular, $g(\lambda)$ is not the zero function. On the other hand, applying Rouche's Theorem to a contour surrounding B_0 forces us to conclude that $g(\lambda)$ has infinitely many roots in a compact set, a contradiction.

The last claim is established by contradiction in a similar manner. Suppose there were a sequence of points $z_j \in S_\delta$ with $|\chi(z_j)| < 1/j$. Chose a sequence of translates $\tau_j \in \mathbb{R}$ so that $z_j - i\tau_j = w_j \in B_0$, and $w_j \to w$. Passing to a subsequence, we create a sequence of functions $f_j(\lambda) = \chi(\lambda + i\tau_j)$ converging uniformly on compact subsets of $\mathbb{C}$ to an entire function $g(\lambda)$. Since $|f_j(w_j)| < 1/j$, $g(w) = 0$.

However, if $f_j(z_j) = 0$, then $|w_j - z_j| \ge \delta$. For j large enough the open ball Ω centered at w with radius $\delta/2$ contains no roots of f_j. The limit function g is not identically 0, so by a theorem of Hurwitz [1, p. 176] $g(\lambda) \ne 0$ for all $\lambda \in \Omega$. ∎

To emphasize the dependence of $\chi(\lambda)$ on T, let

$$\chi_T(\lambda) = \det(T^{-1} - \exp(\lambda W)).$$

The previous lemma will now be used to show that the spectrum of $\mathcal{D}$ may be partitioned into finite systems of eigenvalues [7, p. 368–371], this decomposition remaining valid for small variations of T.

Lemma 3.3. *Fix W and suppose T_0 is invertible. There are numbers $\sigma > 0$ and $\epsilon > 0$, and an integer indexed sequence $\{h_n\}$ with $n < h_n < n+1$, such that for every T with $\|T - T_0\| < \epsilon$,*

$$|\chi_T(\lambda)| \ge \sigma, \quad \Im(\lambda) \in \{h_n\}.$$

Proof. The result is easy to establish outside of some strip $-A \le \Re(\lambda) \le A$. Recall that

$$\chi_T(\lambda) = (-1)^N \exp(\lambda r_0) + \sum c_j \exp(\lambda r_j), \quad r_J < \cdots < r_1 < r_0 = \sum_{n=1}^{N} w_n,$$

with $r_J = 0$ and $c_J = \det(T^{-1})$. If ϵ is sufficiently small then there is a number σ_1 such that $|\det(T^{-1})| \ge \sigma_1 > 0$ for all T with $\|T - T_0\| < \epsilon$. Moreover the coefficients c_j are polynomial functions of the entries of T, so are continuous functions of T. The form of $\chi_T(\lambda)$ shows that there is a number A with

$$|\chi_T(\lambda)| \ge \sigma_1/2, \quad \Re(\lambda) < -A,$$

and since the growth of $\chi_T(\lambda)$ is controlled by $\exp(\lambda r_0)$ as $\Re(\lambda) \to +\infty$,

$$|\chi_T(\lambda)| \ge 1, \quad \Re(\lambda) > A.$$

Next, pick $\delta > 0$, and recall that S_δ is the set of points in the strip S_A whose distance from the roots of $\chi(\lambda) = \chi_{T_0}(\lambda)$ is at least δ. By Lemma 3.2, the number of roots of $\chi(\lambda)$ in a rectangle

$$B_n = \{-A \le \Re(\lambda) \le A, n \le \Im(\lambda) \le n+1\}$$

is bounded independent of n, so if δ is small enough, then every box B_n contains a horizontal line $\Im(\lambda) = h_n$ which is contained in S_δ. Choose $\sigma > 0$ so that $|\chi(\lambda)| \ge 2\sigma$ for $\lambda \in S_\delta$, and in particular for λ on the lines $\Im(\lambda) = h_n$.

Finally, since the coefficients c_j are continuous functions of T, and the functions $\exp(\lambda r_j)$ are bounded in the strip $-A \le \Re(\lambda) \le A$, it is possible to shrink ϵ sufficiently that

$$|\chi_T(\lambda)| \ge \sigma, \quad \Im(\lambda) \in \{h_n\}, \quad \|T - T_0\| < \epsilon.$$

□

3.3 Generic behavior

Among all invertible matrices $T : \mathbb{C}^N \to \mathbb{C}^N$, the ones well matched to a given directed graph will be those given by a collection $\{T(v) : \mathbb{C}^{\deg(v)} \to \mathbb{C}^{\deg(v)}\}$ indexed by the graph vertices and taking incoming boundary data at v to outgoing boundary data at v. We may identify this collection with a single matrix $T : \mathbb{C}^N \to \mathbb{C}^N$ by choosing an ordering of the directed edges. A theorem related to the next result appeared in [2].

Theorem 3.4. *Fix a set of edge weights W. Among the collections of invertible matrices $\{T(v) : \mathbb{C}^{\deg(v)} \to \mathbb{C}^{\deg(v)}\}$ there is a dense set whose corresponding characteristic functions $\chi(\lambda)$ have all roots of multiplicity 1, and whose corresponding operators $\mathcal{D}$ have all eigenvalues of algebraic multiplicity 1.*

Proof. The first step is to show that some collection $\{T_0(v)\}$ has the desired property. Beginning at any vertex, follow the directed edges of $\mathcal{G}$ until a vertex v is repeated. Let γ denote the portion of this path from v to v. Since each vertex of $\mathcal{G}$ has the same number of incoming and outgoing edges, the removal of γ leaves another graph of the same type. In this way, $\mathcal{G}$ may be decomposed into an edge disjoint collection $\{\gamma_i\}$of directed cycles. Pick one vertex $v \in \gamma$ with incoming edge e_j and outgoing edge e_k. The set $\{T_0(v)\}$ may be defined by requiring that for each γ_i and chosen vertex v_i the vertex conditions satisfy $f_k = \mu(j,k) f_j$ at v_i for some complex number $\mu(j,k) \neq 0$, and T is multiplication by 1 at the other vertices of γ_i.

The operator $\mathcal{D}$ may now be viewed as having an associated graph $\tilde{\mathcal{G}}$ which is a decoupled collection of circles. Simple computations give values of $\mu(j,k)$ so that the eigenvalues of $\mathcal{D}$ all have algebraic multiplicity 1, and the characteristic function $\chi(\lambda)$ has all roots of multiplicity 1.

Next, consider a line $\{(1-t)T_0(v) + tT_1(v)\}$ taking the matrix $\{T_0(v)\}$ to $\{T_1(v)\}$. If $\{T_1(v)\}$ is close enough to $\{T_0(v)\}$ the the line will lie in invertible matrices and Lemma 3.3 may be applied to partition the eigenvalues of the path of operators $D(t)$ into finite systems of eigenvalues separated by the lines $\Im(\lambda) = h_n$. The resolvents $(\mathcal{D}(t) - \lambda I)^{-1}$ are analytic operator valued functions of t along the boundary of the rectangles

$$B_n = \{-A \le \Re(\lambda) \le A, h_n \le \Im(\lambda) \le h_{n+1}\}.$$

The discussion in [7, pp. 368-371] may be applied, showing that the eigenvalues in each system all have algebraic multiplicity 1 except for a finite set of exceptional values of t, the exceptional sets depending on n.

We elaborate a bit on the last paragraph, and the analogous argument for the characteristic functions. The sum r_n of the dimensions of the generalized eigenspaces inside each B_n is independent of t, as is the number of roots, counted with multiplicity, of the characteristic functions $\chi_t(\lambda)$. To detect multiple eigenvalues or roots [11, pp. 132-139], consider the function $g(z_1, \dots, z_K)$ of $K = r_n$ complex variables given by

$$g(z_1, \dots, z_K) = \prod_{i \neq j} (z_i - z_j),$$

which is the discriminant of the monic polynomial $p(z)$ with roots $z_1, \dots, z_N$. Obviously g vanishes if and only if two coordinates agree, and g is a symmetric function of the coordinates. It follows that g is a polynomial of the elementary symmetric functions of $z_1, \dots, z_N$, that is the coefficients of $p(z)$. By Newton's identities [5, p. 208] g is a polynomial function of the power sums

$$s_j = z_1^j + \cdots + z_K^j, \quad j = 1, \dots, K.$$

By using traces of Dunford-Taylor integrals [7, p. 44], the power sums $s_j(t)$ for the eigenvalues inside a rectangle B_n are analytic in t. Since the discriminant of the eigenvalues in B_n is nonzero at $t = 0$, it can only vanish at finitely many points, and all eigenvalues have generalized eigenspaces of dimension 1 except for a countable set of t. A similar argument replacing the Dunford-Taylor integral by the Cauchy integral formula applies to the characteristic functions $\chi_t(\lambda)$.

Finally, since the set $Gl(N, \mathbb{C})$ of invertible complex $N \times N$ matrices is path connected, the argument may be extended by piecewise linear paths throughout the entire set.

$\square$

If Γ is a simple closed curve in the resolvent set of $\mathcal{D}_t$ for $0 \leq t \leq 1$, then the sum of the dimensions of the generalized eigenspaces inside Γ is independent of t, as is the number of roots, counted with multiplicity, of the characteristic functions $\chi_t(\lambda)$. Together with the previous result, this gives the following conclusion.

Theorem 3.5. *Fix a set of edge weights W. For any invertible $T : \mathbb{C}^N \to \mathbb{C}^N$ the algebraic multiplicity of an eigenvalue λ_0 of $\mathcal{D}$ is equal to the multiplicity of λ_0 as a root of $\xi_T(\lambda)$.*

3.4 Weyl estimate

Picking a sequence $\{h_n\}$ as in Lemma 3.3, introduce the eigenvalue counting function

$$N(n) = |\{\lambda_k \in spec(\mathcal{D}), h_{-n} < \Im(\lambda_k) < h_n\}|,$$

the eigenvalues counted with algebraic multiplicity.

Theorem 3.6. *For any $p > 0$,*

$$N(n) = \frac{n}{\pi}\sum w_n + O(n^p), \quad n \to \infty.$$

Proof. By Theorem 3.5 it suffices to count roots of the characteristic function $\chi(\lambda)$ using the argument principle [1, p. 151]. From the form (3.6) of $\chi(\lambda)$ there is a constant $\sigma > 0$ such that the following estimates hold for sufficiently large positive A. First,

$$|\chi(\lambda) - \det(T^{-1})| < \exp(\sigma\Re(\lambda)), \quad \Re(\lambda) \le -A. \tag{3.7}$$

Since

$$\chi'(\lambda) = (-1)^N r_0 \exp(\lambda r_0) + \sum_{j<J} c_j r_j \exp(\lambda r_j),$$

it follows that

$$|\chi'(\lambda)/\chi(\lambda)| < \exp(\sigma\Re(\lambda)), \quad \Re(\lambda) \le -A, \tag{3.8}$$

and

$$|\chi'(\lambda)/\chi(\lambda) - r_0| < \exp(-\sigma\Re(\lambda)), \quad \Re(\lambda) \ge A. \tag{3.9}$$

Finally, there is a constant C such that

$$|\chi'(\lambda)/\chi(\lambda)| < C, \quad \Im(\lambda) = h_n. \tag{3.10}$$

Now consider the sequence of rectangular contours $\gamma(n)$ with counter-clockwise orientation, with sides $\gamma_1, \dots, \gamma_4$,

$$\gamma_1 = x + ih_{-n}, \quad -A \le x \le A, \quad \gamma_2 = A + iy, \quad h_{-n} \le y \le h_n,$$

$$\gamma_3 = x + ih_n, \quad -A \le x \le A, \quad \gamma_4 = A - iy, \quad h_{-n} \le y \le h_n.$$

Since χ is entire, the argument principle says that

$$N(n) = \frac{1}{2\pi i}\int_{\gamma(n)} \frac{\chi'(\lambda)}{\chi(\lambda)}\, d\lambda$$

is the number of zeros of χ, counted with multiplicity, inside γ. The main contribution comes from integration over γ_2, with

$$\frac{1}{2\pi i}\int_{\gamma_2}\frac{\chi'(\lambda)}{\chi(\lambda)}\,d\lambda = (h_n - h_{-n})[\frac{r_0}{2\pi} + O(\exp(-\sigma A))].$$

The estimates above give

$$\Big|\frac{1}{2\pi}\int_{\gamma_1}\frac{\chi'(\lambda)}{\chi(\lambda)}\,d\lambda\Big| \le 2AC,$$

and similarly for the integration over γ_3. The contribution from γ_4 is

$$\Big|\frac{1}{2\pi}\int_{\gamma_4}\frac{\chi'(\lambda)}{\chi(\lambda)}\,d\lambda\Big| = O((h_n - h_{-n})\exp(-\sigma A)).$$

Taking $A = (h_n - h_{-n})^p$ for $0 < p < 1$ gives the desired estimate.

$\square$

Deeper results [6] from the theory of almost periodic functions can be used to address the distribution of eigenvalues with respect to $\Re(\lambda)$.

3.5 Generalized eigenfunctions

For nonnormal operators, questions about completeness of generalized eigenfunction and convergence of generalized eigenfunction expansions are usually treated by considering a sequence of spectral projections given by contour integration of the resolvent,

$$P_\gamma f = \frac{-1}{2\pi i}\int_\gamma (\mathcal{D} - \lambda I)^{-1} f\,d\lambda, \tag{3.11}$$

which are [7, pp. 180,181, 187] projections onto the span of the generalized eigenfunctions of $\mathcal{D}$ with eigenvalues contained in γ.

Theorem 3.7. *If f is continuously differentiable with*

$$f^{(k)}(0) = 0 = f^{(k)}(1), \quad k = 0, 1,$$

then there is a sequence of positively oriented simple closed curves $\gamma(n)$ such that

$$\lim_{n\to\infty}\|f(x) - P_{\gamma(n)}f(x)\| = 0.$$

Consequently, the generalized eigenfunctions of $\mathcal{D}$ have dense span in $\mathbb{H}$.

Proof. Given an invertible vertex scattering matrix T_0, let $\mathcal{D}_0$ denote the operator (3.1) with domain defined by T_0. Select a second vertex scattering matrix T_1 such that each vertex map $T_1(v)$ is unitary, so the corresponding $\mathcal{D}_1$ is a skew-adjoint operator [2]. A sequence $\{h_n\}$ may be selected so the properties from Lemma 3.3 are satisfied for both $\mathcal{D}_0$ and $\mathcal{D}_1$. The contours $\gamma(n)$ will be the positively oriented boundary of the rectangular regions given by

$$h_{-n} \le \Im(\lambda) \le h_n, \quad -n \le \Re(\lambda) \le n.$$

Let $P_{\gamma(n)}$ be the spectral projections of (3.11) for $\mathcal{D}_0$ using the contours $\gamma(n)$, and let $\tilde{P}_{\gamma(n)}$ be the analogous sequence of projection operators for $\mathcal{D}_1$.

Letting

$$M_j(\lambda) = (T_j^{-1} - \exp(\lambda W))^{-1}, \quad j = 0, 1,$$

the resolvent formula (3.4) shows that

$$(\mathcal{D}_1 - \lambda I)^{-1} f(x) - (\mathcal{D}_0 - \lambda I)^{-1} f(x)$$

$$= \exp(\lambda x W)\Big[M_1(\lambda) - M_0(\lambda)\Big] \int_0^1 \exp(\lambda[1-t]W) W f(t)\, dt.$$

Suppose each component of f has K continuous derivatives, with

$$f^{(k)}(0) = 0 = f^{(k)}(1), \quad k = 0, \dots, K.$$

Then integration by parts gives

$$\int_0^1 \exp(\lambda[1-t]W) W f(t)\, dt = \lambda^{-K} \int_0^1 \exp(\lambda[1-t]W) W^{-K} W f^{(K)}(t)\, dt.$$

Using Theorem 3.1, for $\lambda \in \gamma(n)$ there are constants $C(K)$ such that

$$\|(\mathcal{D}_1 - \lambda I)^{-1} f(x) - (\mathcal{D}_0 - \lambda I)^{-1} f(x)\|$$

$$\le C(K)|\lambda|^{-K}\Big[\|(\mathcal{D}_1 - \lambda I)^{-1}\| + \|(\mathcal{D}_0 - \lambda I)^{-1}\|\Big]\|f^{(K)}\|$$

$$\le C(K)\frac{|\lambda|^{-K}}{1 + |\Re(\lambda)|}\|f^{(K)}\|,$$

so for $K \ge 1$,

$$\lim_{n\to\infty} \|P_{\gamma(n)} f - \tilde{P}_{\gamma(n)} f\| = 0.$$

Finally,

$$\|f - \lim_{n\to\infty} P_{\gamma(n)} f\| \le \|f - \lim_{n\to\infty} \tilde{P}_{\gamma(n)} f\| + \|\lim_{n\to\infty}[P_{\gamma(n)} f - \tilde{P}_{\gamma(n)} f]\| = 0$$

since $\mathcal{D}_1$ is skew adjoint.

$\square$

4 Rational graphs

Consider as before the operator $\mathcal{D}$ acting componentwise by $w_n^{-1}\partial_x$ on the weighted space $\oplus_n L^2([0,1],w_n)$. In this section the edge weights w_n are assumed to be positive integer multiples of a common value, assumed to be 1 without loss of generality. The assumption of integer weights leads to a considerable strengthening of the results of the previous section; the first reflection of this added structure is that the characteristic function $\chi(\lambda)$ becomes a periodic function of λ.

To probe more deeply into the features implied by integer weights, it is helpful to recast the operator $\mathcal{D}$. First, we employ an edgewise linear change of variables $x = w_n\tilde{x}$ taking $\oplus_n L^2([0,1],w_n)$ to $\oplus_n L^2([0,w_n])$. In these local coordinates the operator acts by ∂_x. Next, add vertices v of degree two at integer distances from the edge endpoints so all edges have length 1, without change of direction. The transition conditions for the new vertices are $f(v^+) = f(v^-)$. These changes have no effect on the domain or action of the differential operator, but they do modify its description upon return to the system description in $\oplus_n L^2[0,1]$. In particular the new number N of components and the size $N \times N$ of the matrix T have increased.

The set of eigenvalues for $\mathcal{D}$ is unchanged, but the characteristic function now has the form

$$\chi(\lambda) = \det(T^{-1} - \exp(\lambda)I_N).$$

The spectrum of $\mathcal{D}$ is the set

$$\exp(\lambda) \in \operatorname{spec}(T^{-1}),$$

making explicit the connection between the spectrum of the operator $\mathcal{D}$ and the set of eigenvalues of a matrix. In particular the spectrum of $\mathcal{D}$ is invariant under translation by $2\pi i$. Suppose $F = [f_1,\dots,f_N]$ is in the domain of $\mathcal{D}$. A simple computation shows that

$$(\partial_x - \lambda - 2\pi i m)\exp(2\pi i m x I_N)F = \exp(2\pi i m x)(\partial_x - \lambda)F.$$

The boundary values of $\exp(2\pi i m x I_N)F$ are the same as those of F. In particular the multiplication map $F \to \exp(2\pi i m x)F$ is an isomorphism from a generalized eigenspace $E(\lambda)$ with eigenvalue λ to the generalized eigenspace $E(\lambda + 2\pi i m)$ with eigenvalue $\lambda + 2\pi i m$.

This discussion is summarized in the next result.

Theorem 4.1. *Suppose the weights w_n for the operator $\mathcal{D}$ are positive integers. Then the spectrum of $\mathcal{D}$ is invariant under translation by $2\pi i$. In the system representation with edge weights* 1*, multiplication by* $\exp(2\pi imx)$ *is an isomorphism from the λ generalized eigenspace to the $\lambda+2\pi im$ generalized eigenspace.*

Having established the previous result, consider an operator $\mathcal{D}_\epsilon = \mathcal{D}$ when the directed graphs $\mathcal{G}$ comes from an undirected graph $\mathcal{G}^u$ by edge splitting as described before (2.14), with the dissipative junction conditions (2.4) and junction scattering matrices (2.10). In this context, the next result highlights an interesting contrast between integral and nonintegral weights.

Theorem 4.2. *Suppose the weights w_n for an operator $\mathcal{D}_\epsilon$ are positive integers. Then the set of eigenvalues of $\mathcal{D}_\epsilon$ includes the sequence $2\pi im$ for $m = 0, \pm 1, \pm 2, \dots$.*

In contrast, assume that $\epsilon(v) < 0$ in the junction conditions (2.4) *for each vertex v of $\mathcal{G}^u$ with* $\deg(v) \geq 2$*. Suppose that for any nontrivial eigenfunction ψ of $\mathcal{D}_\epsilon$ there is a vertex v where $\psi(v) \neq 0$, and that v lies on two cycles in the directed graph $\mathcal{G}$ with the property that the sums of the weights w_n on the cycles are linearly independent over the rationals. Then no eigenvalue for $\mathcal{D}_\epsilon$ has real part zero.*

Proof. Suppose the weights w_n are positive integers. Each edge e of the undirected graph $\mathcal{G}^u$ has an associated impedance Z_e. If $e_1, \dots, e_N$ is a local indexing of the incident edges at a vertex v, Proposition 2.2 shows that the junction scattering matrices $T_\epsilon(v)$ have an eigenvector $[1/\sqrt{Z_1}, \dots, 1/\sqrt{Z_N}]$ with eigenvalue 1. For each directed edge leaving v, define functions with edge values

$$f_n(m,x) = \exp(2\pi imw_nx)/\sqrt{Z_n}, \quad 0 \leq x \leq 1, \quad m = 0, \pm 1, \pm 2, \dots.$$

Extending this construction to all vertices v produces an eigenfunction of $\mathcal{D}_\epsilon$ with eigenvalue $2\pi im$.

To construct graphs so that no eigenvalue for $\mathcal{D}_\epsilon$ has real part zero, begin by assuming that there is a nontrivial eigenfunction ψ for $\mathcal{D}_\epsilon$ with a purely imaginary λ. For each vertex v with in degree at least 2, pick a local indexing of the incoming and outgoing edges, and let $\psi_i(v)$, respectively $\psi_o(v)$, denote the vector of incoming, respectively outgoing, values for ψ. Integration by parts gives

$$0 = \langle \mathcal{D}_\epsilon \psi, \psi \rangle + \langle \psi, \mathcal{D}_\epsilon \psi \rangle$$

$$= \sum_v [\psi_i(v) \cdot \psi_i(v) - \psi_o(v) \cdot \psi_o(v)]$$

$$= \sum_v (I - T_\epsilon^*(v) T_\epsilon(v)) \psi_i(v) \cdot \psi_i(v).$$

By (2.11), if $\epsilon(v) < 0$, then the right hand side can be 0 only if $q_n(v) = 0$ for all edges e_n of $\mathcal{G}^u$ incident on v. This in turn means that $p_m(v) = p_n(v)$. Notice that if $\deg(v) = 1$ in $\mathcal{G}^u$, then the junction scattering matrix is just multiplication by 1. Thus by (2.2) the eigenfunction ψ extends continuously across all vertices.

Let γ_1 be a cycle (simple closed curve) in $\mathcal{G}$, with the edge directions consistent with the direction of the cycle. Since ψ is continuous at each vertex, it is continuous along γ_1. On each directed edge e_n the function ψ is a multiple of $\exp(\lambda w_n x)$. Pick a starting vertex v for the cycle γ_1; if ψ is not the zero function along γ_1, we may assume that $\psi(v) = 1$.

Suppose γ_1 has length $l_1 = w_1 + \cdots + w_K$. Follow ψ around γ_1. The normalization $\psi(v) = 1$, together with the continuity of ψ, means that $\psi(x) = \exp(\lambda w_1 x)$ on the first edge, is $\exp(\lambda w_1) \exp(\lambda w_2 x)$ on the second edge, etc. Upon returning to v we find $1 = \exp(\lambda l_1)$, so for some nonzero integer m_1,

$$l_1 = 2\pi i m_1 / \lambda.$$

Now suppose γ_2 is another cycle starting at v with length l_2 and integer m_2 defined as before. Then

$$l_1/m_1 = l_2/m_2,$$

the cycle lengths are rationally dependent, and the proof is established.

□

Notice in Theorem 4.2 that the conditions implying that no eigenvalue for $\mathcal{D}_\epsilon$ has real part zero will be satisfied if every vertex v of $\mathcal{G}^u$ lies on two independent cycles, and the set of all edge weights w_n for edges e_n of $\mathcal{G}^u$ is independent over the rationals.

Returning to the general directed graphs $\mathcal{G}$ with weights $w_n = 1$, suppose $\lambda_1, \dots, \lambda_K$ are the distinct eigenvalues of $\mathcal{D}$ with $0 \le \Im(\lambda) < 2\pi$. Let $E(\lambda_k)$ denote the corresponding generalized eigenspaces with dimensions $d(k)$ and orthonormal bases $\Psi(\lambda_k) = (\psi_1(\lambda_k), \dots, \psi_{d(k)}(\lambda_k))$. By Theorem 4.1 and Theorem 3.7 the vectors

$$\exp(2\pi i m x I_N) \Psi(\lambda_k), \quad m = 0, \pm 1, \pm 2, \dots, \quad k = 1, \dots, K$$

are linearly independent with dense span in $\mathbb{H}$.

The next lemma describes the linkage between generalized eigenspaces of T and generalized eigenspaces of $\mathcal{D}$.

Theorem 4.3. *Suppose $\mathcal{G}$ has weights $w_n = 1$, λ is an eigenvalue for $\mathcal{D}$, and $\mu = \exp(-\lambda)$. The dimensions of the λ eigenspace for $\mathcal{D}$ and μ eigenspace for T are the same, as are the dimensions of the generalized eigenspace $E(\lambda)$ and the generalized eigenspace for T with eigenvalue μ.*

If $\psi(x,\lambda)$ is in the generalized eigenspace $E(\lambda)$ for $\mathcal{D}$, then there is a K such that

$$\psi = \exp(\lambda x)\sum_{k=0}^{K-1}(1-x)^k F_k, \quad F_{K-1} \ne 0, \tag{4.1}$$

where the constant vectors F_k satisfy

$$(T-\mu I)^j F_{K-j} = 0.$$

Proof. The last claim is established first. Suppose $\psi \in E(\lambda)$ and

$$(\mathcal{D}-\lambda)^K\psi = 0, \quad (\mathcal{D}-\lambda)^{K-1}\psi \ne 0.$$

Any solution of the differential equation $(\partial_x - \lambda)^K\psi = 0$ has the form in (4.1) for some constant vectors F_k. In addition, for $k = 0,\dots,K-1$ the functions

$$\phi_k(x,\lambda) = (\mathcal{D}-\lambda I)^k\psi(x,\lambda)$$

must be in the domain of $\mathcal{D}$, so

$$\phi_k(0,\lambda) = T\phi_k(1,\lambda). \tag{4.2}$$

An induction argument will show that

$$(T-\mu I)^j F_{K-j} = 0.$$

Starting with $j = 1$, note that

$$\phi_{K-1}(x,\lambda) = (-1)^{K-1}(K-1)!\exp(\lambda x)F_{K-1}.$$

The condition (4.2) gives

$$(-1)^{K-1}(K-1)!F_{K-1} = (-1)^{K-1}(K-1)!T\exp(\lambda)F_{K-1},$$

so F_{K-1} is a nonzero eigenvector for T with eigenvalue $\mu = \exp(-\lambda)$.

For larger j,

$$\phi_{K-j} = (\mathcal{D} - \lambda I)^{K-j}\psi$$

$$= (-1)^{K-j} \exp(\lambda x) \sum_{k=K-j}^{K-1} k(k-1)\dots(k+j-K+1)(1-x)^{k+j-K} F_k.$$

The boundary condition $\phi_{K-j}(0) = T\phi_{K-j}(1)$ gives

$$(-1)^{K-j} \sum_{k=K-j}^{K-1} k(k-1)\dots(k+j-K+1)F_k = (-1)^{K-j} T(K-j)! \exp(\lambda) F_{K-j},$$

or

$$(T - \mu I)F_{K-j} = \sum_{k>K-j} c_k F_k, \quad c_k = \mu \frac{k(k-1)\dots(k+j-K+1)}{(K-j)!}. \qquad (4.3)$$

By the induction hypothesis,

$$(T - \mu I)^j F_{K-j} = 0.$$

Next consider the eigenspaces of $\mathcal{D}$ at λ and T at $\mu = \exp(-\lambda)$. So far it is established that if ψ is an eigenfunction of $\mathcal{D}$, then

$$\psi(x, \lambda) = \exp(\lambda x)F$$

for some eigenvector F of T with eigenvalue $\mu = \exp(-\lambda)$. Conversely, if

$$TF = \mu F, \quad \exp(-\lambda) = \mu,$$

define

$$\psi(x, \lambda) = \exp(\lambda x)F.$$

Then $\psi' = \lambda\psi$ and

$$\psi(0, \lambda) = F = \exp(-\lambda)\psi(1, \lambda) = T\psi(1, \lambda),$$

so $\psi(x, \lambda)$ is an eigenvector for $\mathcal{D}$. The map $F \to \exp(\lambda x)F$ is thus an isomorphism of the μ eigenspace of T and the λ eigenspace of $\mathcal{D}$.

To establish the equality of dimensions for generalized eigenspaces, first use Theorem 3.4 to find a sequence of invertible matrices T_n and corresponding operators $\mathcal{D}_n$ such that $T_n \to T$, and all eigenvalues of $\mathcal{D}_n$ have algebraic multiplicity 1. Suppose λ is an eigenvalue of $\mathcal{D}$ with algebraic multiplicity K. Find a small circle γ centered at λ so that λ is the only eigenvalue of $\mathcal{D}$ contained on or within γ, and the characteristic function for $\mathcal{D}_n$ does not vanish on γ for n large enough. For these n there are K eigenvalues of $\mathcal{D}_n$ inside γ. A perturbation of the matrices T_n gives all eigenvalues of T_n algebraic multiplicity 1. Now by continuity of the dimension of the range of the spectral projection associated to γ for $\mathcal{D}$, and the corresponding projection for a contour γ_2 near μ, we see that the dimension of $E(\lambda)$ is the same as the dimension of the generalized eigenspace for T with eigenvalue μ.

$\square$

Recall that a set of vectors $\{\psi_k\}$ in a Hilbert space is a Riesz basis if there is an orthonormal basis $\{\phi_k\}$ and a bounded, boundedly invertible operator U such that $\psi_k = U\phi_k$.

Theorem 4.4. *Suppose $\mathcal{G}$ has rational weights and $\mathcal{D}$ has all eigenvalues simple. Then there is a Riesz basis of eigenvectors of $\mathcal{D}$.*

Proof. We pick a system representation of $\mathcal{D}$ so all edge weights are one, as above. By Theorem 4.3 we can pick a basis $F_1, \dots, F_K$ of eigenvectors for T with eigenvalues $\mu_k = \exp(-\lambda_k)$ and $0 \le \Im(\lambda_k) < 2\pi$ such that a basis of eigenfunctions of $\mathcal{D}$ is given by

$$\psi_{k,n}(x) = \exp((\lambda_k + 2\pi i n)x)F_k.$$

Let $\{E_k\}$ be the standard basis for $\mathbb{C}^K$, written as column vectors, and construct the orthonormal basis $\phi_{k,n} = \exp(2\pi i n x)E_k$ for $\oplus_{k=1}^K L^2[0,1]$. Let $\mathcal{F}$ denote the $K \times K$ matrix whose k-th column is F_k. Then

$$\psi_{k,n} = \mathcal{F}\mathrm{diag}[\exp(\lambda_1 x), \dots, \exp(\lambda_K x)]\phi_{k,n},$$

and the bounded, boundedly invertible operator is multiplication by the matrix function

$$\mathcal{F}\mathrm{diag}[\exp(\lambda_1 x), \dots, \exp(\lambda_K x)].$$

$\square$

References

[1] L. Ahlfors. *Complex Analysis.* McGraw-Hill, New York, 1966.

[2] R. Carlson. Inverse eigenvalue problems on directed graphs. Transactions of the American Mathematical Society, 351(10):4069–88, 1999.

[3] R. Carlson. *Linear Network Models Related to Blood Flow* in Quantum Graphs and Their Applications Contemporary Mathematics 415: 65–80 2006.

[4] Y. Fung, *Biomechanics.* Springer, New York, 1997.

[5] I. Herstein. *Topics in Algebra.* Xerox College Publishing, Waltham, 1964.

[6] B. Jessen and H. Tornhave. *Mean Motions and Zeros of Almost Periodic Functions* Acta Math. 77 (2009), 137–279.

[7] T. Kato. *Perturbation Theory for Linear Operators.* Springer-Verlag, New York, 1995.

[8] V. Kostrykin, J. Potthoff, and R. Schrader. *Contraction semigroups on metric graphs* in Analysis on Graphs and Its Applications PSUM 77: 423–458 2008.

[9] T. Kottos and U. Smilansky. *Periodic orbit theory and spectral statistics for quantum graphs* Ann. Phys. 274: 76–124 1999.

[10] B. Levin. *Distribution of Zeros of Entire Functions.* American Mathematical Society, Providence, 1980.

[11] S. Lang. *Algebra.* Addison-Wesley, 1970.

[12] P. Magnusson, G. Alexander, V. Tripathi and A. Weisshaar. *Transmission Lines and Wave Propagation.* CRC Press, Boca Raton, 2001.

[13] G. Miano and A. Maffucci. *Transmission Lines and Lumped Circuits.* Academic Press, San Diego, 2001.

[14] L.J. Myers and W.L. Capper. *A transmission line model of the human foetal circulatory system* Medical Engineering and Physics 24 (2002), 285–294.

[15] J. Ottesen, M. Olufsen, J. Larsen. *Applied Mathematical Models in Human Physiology.* SIAM, 2004.

[16] S. Sherwin, V. Franke, J. Peiro, K. Parker *One-dimensional modeling of a vascular network in space-time variables.* Journal of Engineering Mathematics **47** (2003), 217–250.

[17] L. Zhou and G. Kriegsmann. A simple derivation of microstrip transmission line equations. *SIAM J. Appl. Math.*, 70(1):353–367, 2009.